Degenerative Coordinates in 22.5° Grid System

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The crease patterns for many origami models are designed within an angular grid system of $90°/n$, for a nonnegative integer n. Precisely, in this system, every (pre)crease passes through an existing reference point in the direction of $m(90/n)°$ for some integer m, and every reference point is either $(0, 0)$, $(1, 0)$, or an intersection of already constructed (pre)creases. For example, 45° ($n = 2$), 30° ($n = 3$), 22.5° ($n = 4$), 18° ($n = 5$), and 15° ($n = 6$) grid systems are known to be useful for the design of origami.

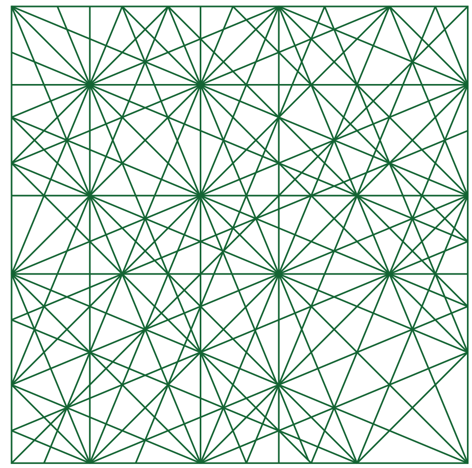

Figure 1: Maekawa-gami: 22.5° grid.

In particular, the 22.5° grid system has been used for centuries—one of the oldest example is the classic origami crane—and the system keeps producing complex but organized origami expressions such as the *Devil* (1980) by Jun Maekawa [1, pp. 146–154] and the *Wolf* (2006) by Hideo Komatsu [2]. Toshikazu Kawasaki calls this system "Maekawa-gami". Figure 1 shows a square filled with several precreases in the 22.5° grid system.

Why are these angular grid systems so useful? A striking feature of Figure 1 is that there are many ways to construct the same point, and as a consequence, many alignments among points and lines. Intuitively, this *degeneracy* of the construction system helps tame the complexity of crease patterns designed within the system.

In this paper, we formalize this notion of degeneracy and organized complexity by characterizing the coordinates of reference points in the 22.5° grid system of the unit square as those points (x, y) with $x, y \in \mathcal{D}_{\sqrt{2}}$, where

$$\mathcal{D}_{\sqrt{2}} = \left\{ x \;\middle|\; x = \frac{m + n\sqrt{2}}{2^\ell} \quad \text{for integers } m, n, \text{and } \ell \geq 0 \right\}.$$

In particular, we establish degeneracy by proving that all constructible points fall into $\mathcal{D}_{\sqrt{2}} \times \mathcal{D}_{\sqrt{2}}$, and establish universality by proving that all points in $\mathcal{D}_{\sqrt{2}} \times \mathcal{D}_{\sqrt{2}}$ can be constructed. In the latter result, the number of required operations is linear in the bit complexities of x and y, where the bit complexity of a number $\frac{m+n\sqrt{2}}{2^\ell} \in \mathcal{D}_{\sqrt{2}}$ is $\lg(1 + |m|) + \lg(1 + |n|) + \ell$.

References

[1] Jun Maekawa. *Genuine Origami*. 2008.

[2] Hideo Komatsu. Wolf. *Origami Tanteidan Magazine*, 17(3):22–32, 2006.

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