Approximation Algorithms via Structural Results for Apex-Minor-Free Graphs

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Abstract. We develop new structural results for apex-minor-free graphs and show their power by developing two new approximation algorithms. The first is an additive approximation for coloring within 2 of the optimal chromatic number, which is essentially best possible, and generalizes the seminal result by Thomassen [32] for bounded-genus graphs. This result also improves our understanding from an algorithmic point of view of the venerable Hadwiger conjecture about coloring H-minor-free graphs. The second approximation result is a PTAS for unweighted TSP in apex-minor-free graphs, which generalizes PTASs for TSP in planar graphs and bounded-genus graphs [20,2,24,15].

We strengthen the structural results from the seminal Graph Minor Theory of Robertson and Seymour in the case of apex-minor-free graphs, showing that apices can be made adjacent only to vortices if we generalize the notion of vortices to "quasivortices" of bounded treewidth, proving a conjecture from [10]. We show that this structure theorem is a powerful tool for developing algorithms on apex-minor-free graphs, including for the classic problems of coloring and TSP. In particular, we use this theorem to partition the edges of a graph into k pieces, for any k, such that contracting any piece results in a bounded-treewidth graph, generalizing previous similar results for planar graphs [24] and bounded-genus graphs [15]. We also highlight the difficulties in extending our results to general H-minor-free graphs.

1 Introduction

Structural graph theory provides powerful tools for designing efficient algorithms in large families of graphs. The seminal work about the structure of graphs is Robertson and Seymour's Graph Minors series of over twenty papers over the past twenty years. From this work, particularly the decomposition theorem for

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graphs excluding any fixed minor H [28], has been made increasingly algorithmic and has led to increasingly general approximation and fixed-parameter algorithms; see, e.g., [14,21,8,7,12,1]. In general, it is interesting to explore how the combinatorial structure of the graph family influences the approximability of classic computational problems.

One such structural graph family that has played an important role in theoretical computer science (e.g., in [17,9,10,7]) is *apex-minor-free graphs*: graphs excluding a fixed apex graph H, where the removal of some vertex of H results in a planar graph. Apex-minor-free graphs include all bounded-genus graphs and many, many more graph families, almost to the extent of general H-minor-free graphs. For example, K_5 is an apex graph, and the class of K_5-minor-free graphs includes $K_{3,n}$ for all n, but the genus of $K_{3,n}$ goes to infinity as n grows.[4] Another example is $K_{3,k}$-minor-free graphs, which according to personal communication were Robertson and Seymour's first step toward their core decomposition result for H-minor-free graphs, because $K_{3,k}$-minor-free graphs can have arbitrarily large genus. More generally, apex-minor-free graphs have all the structural elements of H-minor-free graphs: clique-sums, bounded-genus graphs, apices, and vortices. Thus apex-minor-free graphs serve as an important testbed for algorithmic graph minor theory.

Eppstein [17] showed that apex-minor-free graphs are the largest minor-closed family of graphs that have a property called *bounded local treewidth*. This property has important algorithmic implications: such graphs admit a general family of PTASs following a generalization of Baker's approach for planar graphs [3]. Since this work, apex-minor-free graphs have been studied extensively, in particular in the bidimensionality theory (see [13]), with many algorithmic applications including more general PTASs and subexponential fixed-parameter algorithms [10,7,11,12,9,13]. On the structural side, it has been shown that apex-minor-free graphs have *linear local treewidth*, i.e., every radius-r neighborhood of every vertex has treewidth $O(r)$ [10]. This bound is best possible and substantially improves the bounds on the running of PTASs based on the generalized Baker's approach, from the previous bound of $2^{2^{2^{O(1/\varepsilon)}}} n^{O(1)}$ to the likely best possible bound of $2^{O(1/\varepsilon)} n^{O(1)}$.

To advance our understanding of how structural graph theory impacts approximation algorithms, we develop new such tools for apex-minor-free graphs. In particular, we develop two new decomposition results, strengthening previous results from Graph Minors [28] and from [15], and proving a conjecture from [10]. We use these decompositions to obtain an additive 2-approximation for graph coloring, improving previous results from [14], and to obtain a PTAS for unweighted TSP, generalizing results from [20] and from [15].

Graph coloring. Graph coloring is one of the hardest problems to approximate. In general graphs, the chromatic number is inapproximable within $n^{1-\varepsilon}$ for any

[4] Also, because $K_{3,k}$ has arbitrarily large genus, and $K_{3,k}$ is itself an apex graph, for any genus g there is a k such that $K_{3,k}$ has genus more than g and thus $K_{3,k}$-minor-free graphs include all genus-g graphs.

$\varepsilon > 0$, unless ZPP = NP [18]. Even for 3-colorable graphs, the best approximation algorithm achieves a factor of $O(n^{3/14} \lg^{O(1)} n)$ [5]. In planar graphs, the problem is 4/3-approximable in the multiplicative sense, but more interestingly can be approximated within an additive 1, essentially because all planar graphs are 4-colorable; these approximations are the best possible unless P = NP. In contrast, graphs excluding a fixed minor H (or even graphs embeddable on a bounded-genus surface) are not $O(1)$-colorable for a constant independent of H (or the genus); the worst-case chromatic number is between $\Omega(|V(H)|)$ and $O(|V(H)|\sqrt{\lg |V(H)|})$. Therefore we need different approaches to approximate coloring of such graphs within small factors (independent of H or genus).

In a seminal paper, Thomassen [32] gives an additive approximation algorithm for coloring graphs embeddable on bounded-genus surfaces that is within 2 of optimal. More precisely, for any $k \geq 5$, he gives a polynomial-time algorithm to test k-colorability of graphs embeddable on a bounded-genus surface. Thus, for bounded-genus graphs, we do not know how to efficiently distinguish between 3, 4, and 5 colorability, but we can otherwise compute the chromatic number, and in all cases we can color within an additive 2 of the chromatic number. This result is essentially best possible: distinguishing between 3 and 4 colorability is NP-complete on any fixed surface, and distinguishing between 4 and 5 colorability would require a significant generalization of the Four Color Theorem characterizing 4-colorability in fixed surfaces.

More recently, a 2-approximation to graph coloring has been obtained for the more general family of graphs excluding any fixed minor H [14]. However, additive approximations have remained elusive for this general situation.

The challenge of an improved approximation for H-minor-free graphs is particularly interesting given its relation to Hadwiger's conjecture, one of the major unsolved problems in graph theory. This conjecture states that every H-minor-free graph has a vertex coloring with $|V(H)|-1$ colors. Hadwiger [22] posed this problem in 1943, and proved the conjecture for $|V(H)| \leq 4$. As mentioned above, the best general upper bound on the chromatic number is $O(|V(H)|\sqrt{\lg |V(H)|})$ [26,29]. Thus, Hadwiger's conjecture is not resolved even up to constant factors, and the conjecture itself is only a worst-case bound, while an approximation algorithm must do better when possible.

Here we develop an additive approximation for graph coloring in apex-minor-free graphs, which are between bounded-genus graphs and H-minor-free graphs. We obtain the same additive error of 2 that Thomassen does for bounded-genus graphs:

Theorem 1. *For any apex graph H, there is an additive approximation algorithm that colors any given H-minor-free graph using at most* 2 *more colors than the optimal chromatic number.*

As mentioned above, the additive constant of 2 is essentially best possible. Also, Thomassen's proof method for bounded-genus graphs [32] does not work for apex-minor-free graphs. More precisely, Thomassen's main result in [32] says that, for any $k \geq 6$, there are only finitely many "k-color-critical graphs" embeddable in a fixed surface. Here a graph G is *k-color-critical* if G is not $(k-1)$-colorable, but removing any edge from G makes it $(k-1)$-colorable. However, for any k, there is any apex graph H such that there are infinitely many k-color-critical H-minor-free graphs.[5]

TSP and related problems. The Traveling Salesman Problem is a classic problem that has served as a testbed for almost every new algorithmic idea over the past 50 years. It has been considered extensively in planar graphs and its generalizations, starting with a PTAS for unweighted planar graphs [20], then a PTAS for weighted planar graphs [2], recently improved to linear time [24], then a quasi-polynomial-time approximation scheme (QPTAS) for weighted bounded-genus graphs [19], recently improved to a PTAS [15]. Grohe [21] posed as an open problem whether TSP has a PTAS in general H-minor-free graphs.

We advance the state-of-the-art in TSP approximation by obtaining a PTAS for unweighted apex-minor-free graphs. Furthermore, we obtain a PTAS for minimum c-edge-connected submultigraph[6] in unweighted apex-minor-free graphs, for any constant $c \geq 2$, which generalizes and improves previous algorithms for $c = 2$ on planar graphs [4,6] and for general c on bounded-genus graphs [15].

Theorem 2. *For any fixed apex graph H, any constant $c \geq 2$, and any $0 < \varepsilon \leq 1$, there is a polynomial-time $(1+\varepsilon)$-approximation algorithm for TSP, and for minimum c-edge-connected submultigraph, in unweighted H-minor-free graphs.*

TSP and minimum c-edge-connected submultigraph are examples of a general class of problems called *contraction-closed problems*, where the optimal solution only improves when contracting an edge. Many other classic problems are contraction-closed but not minor-closed, for example, dominating set (and its many variations) and minimum chordal completion. For this reason, contraction-closed problems have been highlighted as particularly interesting in the bidimensionality theory (see [13]).

To obtain the PTASs for TSP and minimum c-edge-connected submultigraph, as well as general family of approximation algorithms for contraction-closed problems, we study a structural decomposition problem introduced in [15]: partition the edges of a graph into k pieces such that contracting any one of the pieces results in a bounded-treewidth graph (where the bound depends on k). Such a result has been obtained for bounded-genus graphs [15] and for planar graphs

[5] Start with any k-color-critical graph G, e.g., K_k. Pick an apex graph H so that G is H-minor-free, which is possible because there is an apex graph of arbitrarily large genus. Now we can combine two vertex-disjoint copies G_1 and G_2 of G into an H-minor-free k-color-critical graph G' using the following *Hajós construction.* Start from the union graph $G_1 \cup G_2$. Pick any edge $\{v_1, w_1\}$ in G_1 and any edge $\{v_2, w_2\}$ in G_2. Remove both of these edges, identify v_1 and v_2, and join w_1 and w_2 by a new edge. The resulting graph G' is H-minor-free and k-color-critical provided G is. By repeating this construction starting from G', etc., we obtain infinitely many k-color-critical H-minor-free graphs.

[6] This problem allows using multiple copies of an edge in the input graph—hence sub*multi*graph—but the solution must pay for every copy.

with a variation of contraction called compression (deletion in the dual graph) [24,25]. These results parallel similar decomposition results for edge deletions in planar graphs [3], apex-minor-free graphs [17], and H-minor-free graphs [16,14]. However, contraction decomposition results are not known for graphs beyond bounded genus.

In this paper, we prove such a contraction decomposition result for apex-minor-free graphs:

Theorem 3. *For any fixed apex graph H, any integer $k \geq 2$, and every H-minor-free graph G, the edges of G can be partitioned into k sets such that contracting any one of the sets results in a graph of treewidth at most $f(k, H)$. Furthermore, such a partition can be found in polynomial time.*

In [15], it is shown how Theorem 3 leads to a general family of PTASs for any contraction-closed problem satisfying a few simple criteria, including TSP and minimum c-edge-connected submultigraph, thus proving Theorem 2.

Structural results. In order to prove both Theorem 1 about coloring approximation and Theorem 3 about contraction decomposition, we need to strengthen the structural results from the seminal Graph Minor Theory in the case of apex-minor-free graphs. Roughly speaking, Robertson and Seymour [28] prove that every H-minor-free graph is a clique sum of graphs "almost embeddable" into bounded-genus surfaces, with the exception of a bounded number of "apex" vertices and a bounded number of "local areas of non-planarity", called "vortices", which have bounded pathwidth. See [14] for the relevant definitions. More recently, this result was made algorithmic [14]. We prove that, when H is an apex graph, the apex vertices can be constrained to have edges only to vertices of vortices, but we have to generalize vortices to what we call "quasivortices", which have bounded treewidth instead of pathwidth. Our result is also algorithmic:

Theorem 4. *For any fixed apex graph H, there is a constant h such that any H-minor-free graph can be written as a clique-sum of h-almost embeddable graphs such that the apex vertices in each piece are only adjacent to quasivortices. Moreover, apices in each piece are not involved in the surface part of other pieces. Furthermore, there is a polynomial-time algorithm to construct this clique-sum decomposition for a given H-minor-free graph.*

Let us observe that it is obviously necessary for the running time of the coloring algorithm to depend exponentially on the excluded apex graph H.

This theorem is a powerful tool for the design of approximation algorithms in apex-minor-free graphs, as we show here for graph coloring and TSP. By analogy, the structural result for H-minor-free graphs has already proved critical throughout graph algorithms—see, e.g., [21,8,12,14,1]—and the additional structure we establish for apex-minor-free graphs seems essential. Indeed, this theorem was conjectured in the context of proving that apex-minor-free graphs have linear local treewidth [10], where it was suggested that this theorem would make it substantially easier to prove linear local treewidth and thereby improve the running time of many PTASs from $2^{2^{2^{O(1/\varepsilon)}}} n^{O(1)}$ to $2^{O(1/\varepsilon)} n^{O(1)}$. In fact, the conjecture of [10] used the standard notion of vortices, and one of our insights is to introduce quasivortices, which are just as good for algorithmic purposes; there is evidence that the use of quasivortices in Theorem 4 is necessary.

In Section 3, we describe the significant difficulties in generalizing our results to general H-minor-free graphs, as we crucially rely on our structure theorems for apex-minor-free graphs.

2 Overview

In this section, we give overviews of the proofs and algorithms behind our three main results. We start in Section 2.1 with our structural theorem about apex-minor-free graphs, which strengthens the Robertson-Seymour clique-sum decomposition. The structure determined by this theorem can be computed in polynomial time, and forms the foundation for our other results. The most direct use is our additive 2-approximation for coloring apex-minor-free graphs, described in Sections 2.2–2.3, which shows how to combine Thomassen's bounded-genus techniques over the new structure of apex-minor-free graphs. The PTASs for unweighted TSP and related problems are based on the contraction decomposition result, as described in Section 2.4, which in turn is based on the new structure of apex-minor-free graphs.

2.1 Overview of Structure Theorem for Apex-Minor-Free Graphs

Our main structure theorem, Theorem 4, builds upon the seminal Graph Minor decomposition theorem together with a new technique that is also developed in the Graph Minors series. The main challenge for our structure theorem is to control the neighborhood of apex vertices. How do we do it? Consider an apex v in the apex set. If v is adjacent to a vertex set W in the surface part in such a way that each vertex in W is far apart from other vertices in W, then we should be able to find a desired apex graph minor, because one of lemmas in Graph Minors tells us that, if a given vertex set is located far apart from each other on a planar graph or a graph on a fixed surface with sufficiently large representativity, then we can find any desired "rooted" planar graph minor. By choosing v to be the apex vertex of H, if there is a rooted $H - v$ minor in the surface part of some piece, we find that the graph actually has an H minor.

Otherwise, the neighbors of each apex vertex in the surface part of each piece are covered by a bounded number of bounded-radius disks. Because the number of apex vertices in each piece is bounded, this implies that there are bounded number of bounded-radius disks in the surface part such that these disks take care of all the neighbors of apices in the surface part of each piece. Now these bounded-radius disks become quasivortices after some modification. More precisely, these disks can decompose into a linear decomposition such that the intersection of two consecutive bags has bounded size. Then we can decompose each bag to extend our structure in such a way that each apex vertex is adjacent only to quasivortices.

This result is quite powerful. For example, it follows that apices in each piece are not involved in the surface part of other pieces. This is because there are no ≤ 3-separations in the surface part (excluding vortices) that have a neighbor in the apex vertex set. Otherwise, we could find a "neighbor" of some apex vertex v by finding a path from v to the surface part through the component. So the surface part could involve a clique-sum, but this is really a ≤ 3-separation. This property helps a lot in our algorithm.

2.2 Overview of Coloring Algorithm

Next we turn to our coloring algorithm. As pointed out before, Thomassen's proof method in [32] does not work for H-minor-free graphs for a fixed apex graph H, because there are infinitely many k-color-critical graphs without H minors for a fixed apex graph H. Nonetheless, some of the results proved in [32] are useful for us. Let us point out that Thomassen's result [32] depends on many other results, mostly by Thomassen himself; see [31,30,33,27]. The result in [32] is considered by many to be one of the deepest results in chromatic graph theory. This is because the series of results obtained by Thomassen opens up how topological graph theory can be used in chromatic graph theory.

At a high level, by our structure theorem, we have a clique-sum decomposition such that each torso (intersection of two pieces) in the surface part involves at most three vertices (in the surface), and no other vertices at all. Because there are at most three vertices in the intersection of pieces in the surface part, we can focus on each bag individually, and the coloring of each bag can be matched nicely by putting cliques in the intersections of two pieces. Let us observe that the clique size here is at most three.

Hence we can really focus on one piece, which has h-almost embeddable structure without any 3-separations in the surface part. Call this graph G. Roughly, what we will do is to decompose G into two parts V_1 and V_2 such that V_1 has bounded treewidth, and V_2 is a union of bounded-genus graphs. We can find such a decomposition such that the boundary of V_1 in V_2 is, roughly, the vertices on the cuffs to which quasivortices are attached. We shall add these boundary vertices of V_2 to V_1, and let V_1' be the resulting graph obtained from V_1. It turns out to be possible to prove that V_1' also has bounded treewidth. To find such a partition, we use the properties of our structure theorem for apex-minor-free graphs.

Next, we color $V_1 \cup V_2$. It is well-known that we can color graphs of bounded treewidth in polynomial time. So we can color V_1' using at most $\chi(G)$ colors. The main challenge is how to extend the precoloring of the vertices in $V_1 \cap V_2$ to the rest of vertices in V_2. To achieve this, we shall need some deep results by Thomassen [32], as described in the next subsection.

2.3 Coloring Extensions in Bounded-Genus Graphs

As we pointed out above, we shall decompose a given graph G into two parts V_1 and V_2 (possibly with $V_1 \cap V_2 \neq \emptyset$) such that V_1 has bounded treewidth, V_2 is union of bounded-genus graphs, and $V_1 \cap V_2$ is, roughly, the vertices on the cuffs to which quasivortices are attached. So we can color the vertices in V_1 using the bounded-treewidth method. Then the vertices in $V_1 \cap V_2$ are precolored.

Our challenge is the following. Suppose the vertices in the bounded number of cuffs are precolored. Can we extend this precoloring to the whole surface part? To answer this question, we use the following tool developed by Thomassen [32]. In fact, our statement below is different from the original by Thomassen. The list-coloring version of the following theorem is proved in [23]. So this immediately implies Theorem 5. But if we only need graph-coloring version of the result, let us point out that it follows from the same proof as in [32] by combining with the metric of Robertson and Seymour.

Theorem 5. *For any two nonnegative integer g, q, d, there exists a natural number $r(g, q, d)$ such that the following holds: Suppose that G is embedded on a fixed surface S of genus g and of the representativity at least $r(g, q, d)$ and there are d disjoint cuffs $S_1, S_2, \ldots, S_d$ such that the distance (in a sense of Robertson and Seymour's metric) of any two cuffs of $S_1, S_2, \ldots, S_d$ is at least q. Suppose furthermore that all the vertices in $S_1, S_2, \ldots, S_d$ are precolored with at most five colors such that*

1. *all the faces except for $S_1, S_2, \ldots, S_d$ are triangles, and*
2. *no vertex v of $G - (S_1 \cup S_2 \cup \cdots \cup S_d)$ is joined to more than two colors unless v has degree 4 or v has degree 5 and v is joined to two vertices of the same color.*

Then the precoloring of $S_1, S_2, \ldots, S_d$ can be extended to a 5-coloring of G. Also, there is a polynomial time algorithm to 5-color G.

For the reader's convenience, let us make some remarks for the proof of Theorem 5. The above statement follows from Theorem 8.1 (by putting $p = 0$) of [32]. The large distance of the metric of Robertson and Seymour implies the existence of large number of canonical cycles in the statement of Theorem 8.1. Then we can extend the result of Theorem 8.1 to the above statement. But as we pointed out above, there is now a stronger theorem, which is the list-coloring version of Theorem 5, see [23].

By a small modification to the theorem above, we obtain the following theorem which we will use:

Theorem 6. *For any two nonnegative integer g, q, d, there exists a natural number $r(g, q, d)$ such that the following holds: Suppose that G is embedded on a fixed surface S of genus g and of the representativity at least $r(g, q, d)$ and there are d disjoint cuffs $S_1, S_2, \ldots, S_d$ such that the distance (in a sense of Robertson and Seymour's metric) of any two cuffs of $S_1, S_2, \ldots, S_d$ is at least q. Suppose furthermore that all the vertices in $S_1, S_2, \ldots, S_d$ are precolored such that*

1. *all the faces except for $S_1, S_2, \ldots, S_d$ are triangles, and*
2. *no vertex v of $G-(S_1 \cup S_2 \cup \ldots \cup S_d)$ is joined to more than $\chi(G)-1$ colors unless v has degree 4 or v has degree 5 and v is joined to two vertices of the same color.*

Then the precoloring of $S_1, S_2, \ldots, S_d$ using at most $\chi(G)$ colors can be extended to a $(\chi(G)+2)$-coloring of G. In fact, in the surface part, we only need five colors for most of vertices. (Precoloring may use more than five colors and vertices that are adjacent to precolored vertices may need some other color, though.) Also, there is a polynomial time algorithm for such a coloring G.

The list-coloring version of Theorem 6 is also proved in [23]. So this immediately implies Theorem 6, but for the reader's convenience, let us make some remarks. By the assumption of Theorem 6, each vertex not on the cuffs $S_1, S_2, \ldots, S_d$ has a list with at least 5 colors available, and each vertex that has a neighbor in the cuffs $S_1, S_2, \ldots, S_d$ has a list with at least 3 colors available. If we delete all the cuffs $S_1, S_2, \ldots, S_d$ in Theorem 5, then each vertex that has a neighbor in the cuffs $S_1, S_2, \ldots, S_d$ has a list with at least 3 colors available. So it follows that the conditions in Theorem 6 are equivalent to that in Theorem 5. Hence Theorem 6 follows from Theorem 5.

Let us observe that the coloring of Theorem 6 may use more than five colors because the precoloring vertices may use $\chi(G)$ colors. On the other hand, most of the vertices in the surface part use only five colors. The exceptional vertices are the precolored vertices on the cuffs, and vertices that are adjacent to precolored vertices.

2.4 Contraction Decomposition Result and PTASs

Finally, we sketch our proof of the contraction decomposition result and its applications to PTASs.

Our proof of Theorem 3 heavily depends on our decomposition theorem. Let us focus on one piece that has an h-almost embeddable structure without any 3-separations in the surface part. Call this graph G. As we did in our coloring algorithm, we decompose the graph G into two parts V_1 and V_2 such that V_1 has bounded treewidth, and V_2 is union of bounded-genus graphs. We can find such a decomposition such that the boundary of V_1 in V_2 is, roughly, the vertices on the cuffs to which quasivortices are attached. We shall add these boundary vertices of V_2 to V_1, and let V_1' be the resulting graph obtained from V_1. Again we can prove that V_1' also has bounded treewidth and V_2' is union of bounded-genus graphs.

Now our goal is to label the edges of the graph such that contracting any label set results in a bounded-treewidth graph. One challenge is the following. Suppose that the edges in the bounded number of cuffs are prelabeled. Can we extend this prelabeling to the whole surface part? Fortunately, this extension can be done by pushing the proof of the result in [15] just a little.

The main challenge is in handling the clique-sums. As we mentioned in the context of the coloring algorithm, we have a clique-sum decomposition such that each torso (intersection of two pieces) in the surface part involves at most three vertices, and nothing else. Because there are at most three vertices in the intersection of two pieces in the surface part, so for our coloring algorithm, we could really focus on each piece separately, and combine the colorings of each piece nicely by putting cliques in the intersection of two pieces. But for the contraction decomposition, this is a serious issue. Fortunately, the clique-sum in the surface part involves at most three vertices. We now put cliques in the intersection of two pieces (i.e., we add all the possible edges in the intersection of two pieces) and precolor these edges. What we need are edge-disjoint paths connecting these at most three vertices such that each path has the same label as the prelabeling of the corresponding edge. This is possible because we only need to control at most three edge-disjoint paths. All we need is to modify the argument in [15] to prove this, and the proof is identical to that in [15]. The details will be provided in the appendix.

As described in [15], the contraction decomposition result of Theorem 3 is strong enough to obtain a PTAS for TSP, minimum c-edge-connected submultigraph, and a variety of other contraction-closed problems, for an unweighted apex-minor-free graph, in particular proving Theorem 2. But it seems very hard to extend this result to H-minor-free graphs, because we do not know how many vertices are involved in the clique-sum—the number is bounded, but it seems to us that the precise number of vertices involved in the clique-sum is important—so we may not be able to control the neighbors of each apex vertex.

3 Difficulties with H-Minor-Free Graphs

One natural question is whether it is possible to extend our approach in this paper to H-minor-free graphs. More specifically, Robin Thomas (private communication) asked whether there is an additive c-approximation for chromatic number in H-minor-free graphs, where c is independent of H. One obvious way to attack this question is to prove the following conjecture:

Conjecture 1. Every H-minor-free graph has a partition of vertices into two vertex sets V_1 and V_2 such that V_1 has bounded treewidth and V_2 has chromatic number at most c for some absolute constant c.

We could add some moderate connectivity condition on the conjecture.

Our approach clearly breaks down for general H-minor-free graphs. Let us highlight some technical difficulties.

1. We need to consider separations of huge order, dependent on H instead of an absolute constant like 3.
2. We can no longer control the neighbors of apices: they can be all over the surface part of the piece.
3. Clique-sums become problematic. In particular, clique-sums involving vertices in the surface part are difficult to handle because of so-called *virtual edges*: edges present in the pieces but not in the clique-sum (the actual

graph). Many pieces may be clique-summed to a common piece, making all of the surface edges in that piece virtual, effectively nonexistent.

So far we have been unable to surmount any of these difficulties, which is why apex-minor-free graphs seems like a natural limiting point.

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