BIDIMENSIONAL PARAMETERS AND LOCAL TREEWIDTH [*]

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Abstract. For several graph-theoretic parameters such as vertex cover and dominating set, it is known that if their sizes are bounded by k then the treewidth of the graph is bounded by some function of k. This fact is used as the main tool for the design of several fixed-parameter algorithms on minor-closed graph classes such as planar graphs, single-crossing-minor-free graphs, and graphs of bounded genus. In this paper we examine the question whether similar bounds can be obtained for larger minor-closed graph classes, and for general families of graph parameters including all those for which such behavior has been reported so far. Given a graph parameter P, we say that a graph family $\mathcal{F}$ has the *parameter-treewidth property for* P if there is an increasing function t such that every graph $G \in \mathcal{F}$ has treewidth at most $t(P(G))$. We prove as our main result that, for a large family of graph parameters called *contraction-bidimensional*, a minor-closed graph family $\mathcal{F}$ has the parameter-treewidth property if $\mathcal{F}$ has bounded local treewidth. We also show "if and only if" for some graph parameters, and thus this result is in some sense tight. In addition we show that, for a slightly smaller family of graph parameters called *minor-bidimensional*, all minor-closed graph families $\mathcal{F}$ excluding some fixed graphs have the parameter-treewidth property. The contraction-bidimensional parameters include many domination and covering graph parameters such as vertex cover, feedback vertex set, dominating set, edge-dominating set, q-dominating set (for fixed q). We use our theorems to develop new fixed-parameter algorithms in these contexts.

Key words. Treewidth, local treewidth, graph minors, dominating set

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1. Introduction.

The last ten years has witnessed the rapid development of a new branch of computational complexity, called parameterized complexity; see the book of Downey & Fellows [19]. Roughly speaking, a parameterized problem with parameter a nonnegative integer k is *fixed-parameter tractable (FPT)* if it admits an algorithm with running time $h(k)\,|I|^{O(1)}$. (Here h is a function depending *only* on k and $|I|$ is the size of the instance.)

A celebrated example of a fixed-parameter tractable problem is Vertex Cover, asking whether an input graph has at most k vertices that are incident to all its edges. When parameterized by k, the k-Vertex Cover problem admits a solution as fast as $O(kn + 1.285^k)$ [9]. Moreover, if we restrict k-Vertex Cover to planar graphs then it is possible to design FPT-algorithms where the contribution of k in the non-polynomial part of their complexity is subexponential. The first algorithm of this type was given by Alber et al. (see [4]). Recently, Fomin and Thilikos reported a $O(k^4 + 2^{4.5\sqrt{k}} + kn)$ algorithm for planar k-Vertex Cover [25].

However, not all parameterized problems are fixed-parameter tractable. A typical

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example of such a problem is DOMINATING SET, asking whether an input graph has at most k vertices that are adjacent to the rest of the vertices. When parameterized by k, the k-DOMINATING SET problem is known to be $W[2]$-complete and thus it is not expected to be fixed-parameter tractable [19]. Interestingly, the fixed-parameter complexity of the same problem can be distinct for special graph classes. During the last five years, there has been substantial work on fixed-parameter algorithms for solving the k-DOMINATING SET on planar graphs and different generalizations of planar graphs. For this class the problem can be solved in $O(8^k n)$ time [2]. An algorithm with a sublinear exponent for the problem with running time $O(4^{6\sqrt{34k}}n)$ was given by Alber et al. [1]. Recently, Kanj & Perković [30] improved the running time to $O(2^{27\sqrt{k}}n)$ and Fomin & Thilikos to $O(2^{15.13\sqrt{k}}k+n^3+k^4)$ [23, 25]. The fixed-parameter algorithms for extensions of planar graphs like bounded-genus graphs and graphs excluding single-crossing graphs as minors are introduced in [13, 15, 20].

In the majority of these results, the design of FPT algorithms for solving problems such as k-VERTEX COVER or k-DOMINATING SET in a sparse graph class $\mathcal{F}$ is based on the following lemma: every graph G in $\mathcal{F}$ where the value of the graph parameter is at most k has treewidth bounded by $t(k)$, where t is a strictly increasing function depending only on $\mathcal{F}$. With some work (sometimes very technical), a tree decomposition of width $t(k)$ is constructed and standard dynamic-programming techniques on graphs of bounded treewidth are implemented. Of course this method can not be applied for any graph class $\mathcal{F}$. For instance, the n-vertex complete graph K_n has a dominating set of size one and treewidth equal to $n-1$. So the emerging question is:

> For which (larger) graph classes and for which graph parameters can
> the "bounding treewidth method" be applied?

In this paper we give a *complete* characterization of minor-closed graph families for which the aforementioned "bounding treewidth method" can be applied for a wide family of graph parameters. For a given graph parameter P, we say that a graph family $\mathcal{F}$ has the *parameter-treewidth property* for P if there is a strictly increasing function $t : \mathbb{N} \to \mathbb{N}$ such for every graph $G \in \mathcal{F}$ where $P(G) \leq k$ implies that G has treewidth at most $t(k)$. For example, it is known [1, 23, 30] that any planar graph with a dominating set of size at most k has treewidth $O(\sqrt{k})$. Therefore, the class of planar graphs has the parameter-treewidth property for the dominating-set parameter.

Our main result is that for a large family of graph parameters called *contraction-bidimensional*, a minor-closed graph family $\mathcal{F}$ has the parameter-treewidth property if $\mathcal{F}$ has bounded local treewidth. Moreover, we show that the inverse is also correct if some simple condition is satisfied by P. In addition we show that, for a slightly smaller family of graph parameters called *minor-bidimensional*, every minor-closed graph family $\mathcal{F}$ excluding some fixed graph has the parameter-treewidth property. The bidimensional-parameter family includes many domination and covering graph parameters such as vertex cover, feedback vertex set, dominating set, edge-dominating set, and q-dominating set (for fixed q) (see also [15, 12] for more examples). Another example of a contraction-bidimensional parameter is the length of a minimum TSP (Traveling Salesman) tour, i.e., the smallest number of edges in a walk visiting all vertices of the graph.

The proof of the main result uses the characterization of Eppstein for minor-closed families of bounded local treewidth [21] and Diestel et al.'s modification of the Robertson & Seymour excluded-grid-minor theorem [18]. In addition, the proof is constructive and can be used for constructing fixed-parameter algorithms to decide bidimensional graph parameters on minor-closed families of bounded local treewidth. These algorithms parallel the general fixed-parameter algorithm of Frick and Grohe [27] for properties definable in first-order logic in graph families of bounded local treewidth; our results apply e.g. to minor-bidimensional parameters definable in monadic second-order logic in nontrivial minor-closed graph families. See Section 5 for details.

This paper is organized as follows. Section 2 contains the formal definitions of the concepts used in the paper. Section 3 presents two combinatorial results which support the main result of the paper, proved in Section 4. Finally, in Section 5 we present the algorithmic consequences of our results and we conclude with some open problems.

2. Definitions and preliminary results. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We let n denote the number of vertices of a graph when it is clear from context. For every nonempty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. We define the *q-neighborhood* of a vertex $v \in V(G)$, denoted by $N_G^q[v]$, to be the set of vertices of G at distance at most q from v. Notice that $v \in N_G^q[v]$. We put $N_G[v] = N_G^1[v]$. We also often say that a vertex v *dominates* subset $S \subseteq V(G)$ if $N_G[v] \supseteq S$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge contractions is said to be a *contraction* of G. We use the notation $H \preceq_c G$ for H a contraction of G. Notice that the relation $H \preceq_c G$ partitions the edge set of G into edges that are also the edges of H and the contracted edges. We say that a vertex v of G is contracted to a vertex u of H if there is a path from u to v in G such that all edges in this path are contracted. A graph H is a *minor* of a graph G if H is the subgraph of a contraction of G. We use the notation $H \preceq G$ [resp. $H \preceq_c G$] for H a minor [a contraction] of G. A family (or class) of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. A minor-closed graph family $\mathcal{F}$ is *H-minor-free* if $H \notin \mathcal{F}$.

The $m \times m$ grid is the graph on $\{1, 2, \ldots, m^2\}$ vertices $\{(i, j)\colon 1 \le i, j \le m\}$ with the edge set

$$\{(i, j)(i', j')\colon |i - i'| + |j - j'| = 1\}.$$

For $i \in \{1, 2, \ldots, m\}$ the vertex set (i, j), $j \in \{1, 2, \ldots, m\}$, is referred as the *$i$th row* and the vertex set (j, i), $j \in \{1, 2, \ldots, m\}$, is referred to as the *ith column* of the $m \times m$ grid. The vertices (i, j) of the $m \times m$ grid with $i \in \{1, m\}$ or $j \in \{1, m\}$ are called *boundary* vertices and the rest of the vertices are called *non-boundary* vertices.

The notion of treewidth was introduced by Robertson and Seymour [31]. A *tree decomposition* of a graph G is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, with $\{X_i \mid i \in I\}$ a family of subsets of $V(G)$ and T a tree, such that

1. $\bigcup_{i \in I} X_i = V(G)$;
2. for all $\{v, w\} \in E(G)$, there is an $i \in I$ with $v, w \in X_i$; and
3. for all $i_0, i_1, i_2 \in I$, if i_1 is on the path from i_0 to i_2 in T, then $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The treewidth $\mathbf{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G.

We need the following facts about treewidth. The first fact is trivial.

- For any complete graph K_n on n vertices, $\mathbf{tw}(K_n) = n - 1$.

The second fact is well known but its proof is not trivial. (See e.g., [17].)

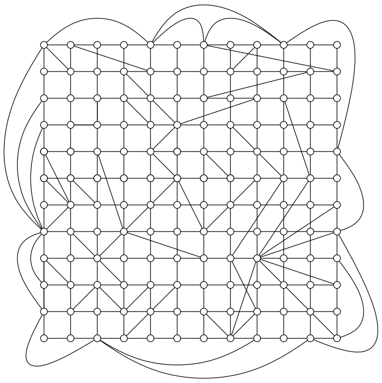

FIG. 2.1. *An augmented* 12×12 *grid with span 8.*

- The treewidth of the $m \times m$ grid is m.

The next fact we need is the improved version of the Robertson & Seymour theorem on excluded grid minors [32] due to Diestel et al. [18]. (See also the textbook [17].)

THEOREM 2.1 ([18]). *Let r, m be integers, and let G be a graph of treewidth at least $m^{4r^2(m+2)}$. Then G contains either K_r or the $m \times m$ grid as a minor.*

Formally, a *graph parameter P* is a function mapping graphs to nonnegative integers. The *parameterized problem associated with P* asks, for a fixed k, whether $P(G) \leq k$ for a given graph G. Given a graph parameter P, we say that a graph family $\mathcal{F}$ has the *parameter-treewidth property for P* if there is a strictly increasing function t such that every graph $G \in \mathcal{F}$ has treewidth at most $t(P(G))$.

DEFINITION 2.2. *Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. We say that a graph parameter P is g-*minor-bidimensional[1] *if*

- *Contracting an edge, deleting an edge, or deleting a vertex in a graph G cannot increase $P(G)$.*
- *For the $r \times r$ grid R, $P(R) \geq g(r)$.*

*Similarly, a graph parameter P is g-*contraction-bidimensional *if*

- *Contracting an edge in a graph G cannot increase $P(G)$.*
- *For any $r \times r$ augmented grid R of constant span, $P(R) \geq g(r)$.*

In the above definition, an $r \times r$ *augmented grid of span s* is an $r \times r$ *grid* with some extra edges such that each vertex is attached to at most s non-boundary vertices of the grid (see an example in Figure 2.1). Intuitively, bidimensional parameters are required to be "large" in two-dimensional grids.

We note that a g-minor-bidimensional parameter is also a g-contraction-bidimensional parameter. One can easily observe that many graph parameters such as minimum sizes of dominating set, q-dominating set (distance q-dominating set for a fixed q), vertex cover, feedback vertex set, and edge-dominating set (see exact definitions of the corresponding graph parameters in [15]) are $\Theta(r^2)$-minor- or $\Theta(r^2)$-contraction-bidimensional parameters.

Here, we present a theorem for minor-bidimensional parameters on general minor-closed classes of graphs excluding some fixed graphs, which plays an important role in the main result of this paper.

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[1] Closely related notions of bidimensional parameters are introduced by the authors in [13].

Theorem 2.3. *If a g-minor-bidimensional parameter P on an H-minor-free graph G has value at most k, then* $\mathbf{tw}(G) \le 2^{4|V(H)|^2(g^{-1}(k)+2)\log(g^{-1}(k))} = 2^{O(g^{-1}(k)\log(g^{-1}(k)))}$.

Proof. Notice that $K_{|V(H)|}$ contains as a minor any graph on $|V(H)|$ vertices. Therefore we may assume that G is $K_{|V(H)|}$-minor-free. If the claimed upper bound for the treewidth of G is not correct, then Theorem 2.1 implies that G contains a $m \times m$ grid R as a minor for $m > g^{-1}(k)$. Because P is g-minor-bidimensional, $P(R) \ge g(m)$. The bidimensionality of P along with the fact that R is a minor of G yield $P(G) \ge g(m)$. Therefore, $k \ge g(m)$, a contradiction. ☐

Theorem 2.3 can be applied for minor-bidimensional parameters such as vertex cover or feedback vertex set.

The notion of local treewidth was introduced by Eppstein [21] (see also [29]). The *local treewidth* of a graph G is

$$\mathbf{ltw}(G,r) = \max\{\mathbf{tw}(G[N_G^r[v]])\colon v \in V(G)\}.$$

For a function $f\colon \mathbb{N} \to \mathbb{N}$ we define the minor-closed class of graphs of bounded local treewidth

$$\mathcal{L}(f) = \{G\colon \forall H \preceq G\ \forall r \ge 0,\ \mathbf{ltw}(H,r) \le f(r)\}.$$

Also we say that a minor-closed class of graphs $\mathcal{C}$ has bounded local treewidth if $\mathcal{C} \subseteq \mathcal{L}(f)$ for some function f.

Well-known examples of minor-closed classes of graphs of bounded local treewidth are graphs of bounded treewidth, planar graphs, graphs of bounded genus, and single-crossing-minor-free graphs.

Many difficult graph problems can be solved efficiently when the input is restricted to graphs of bounded treewidth (see e.g., Bodlaender's survey [7]). Eppstein [21] made a step forward by proving that some problems like subgraph isomorphism and induced subgraph isomorphism can be solved in linear time on minor-closed graphs of bounded local treewidth. Also the classic Baker's technique [6] for obtaining approximation schemes on planar graphs for different NP-hard problems can be generalized to minor-closed families of bounded local treewidth. (See [21] for a generalization of these techniques.)

An *apex graph* is a graph G such that, for some vertex v (the *apex*), $G - v$ is planar. The following result is due to Eppstein [21].

Theorem 2.4 ([21]). *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ is of bounded local treewidth if and only if $\mathcal{F}$ does not contain all apex graphs.*

3. Combinatorial lemmas.

In this section we prove two combinatorial lemmas regarding grids and graphs of bounded local treewidth.

Lemma 3.1. *Suppose we have an $m \times m$ grid H and a subset S of vertices in the central $(m-2\ell) \times (m-2\ell)$ subgrid H', where $s = |S|$ and $\ell = \lfloor \sqrt[4]{s} \rfloor$. Then H has as a minor the $\ell \times \ell$ grid R such that each vertex in R is a contraction of at least one vertex in S and other vertices in H.*

Proof. Let s_x denote the number of distinct x coordinates of the vertices in S, and let s_y denote the number of distinct y coordinates of the vertices in S. Thus, $s \le s_x \cdot s_y$. Assume by symmetry that $s_y \ge s_x$, and therefore $s_y \ge \sqrt{s}$.

We define the subset S' of S by removing all but one point that share a common y coordinate, for each y coordinate. Thus, all y coordinates of the vertices in S' are distinct, and $|S'| = s_y$. We discard all but ℓ^2 vertices in S' to form a slightly

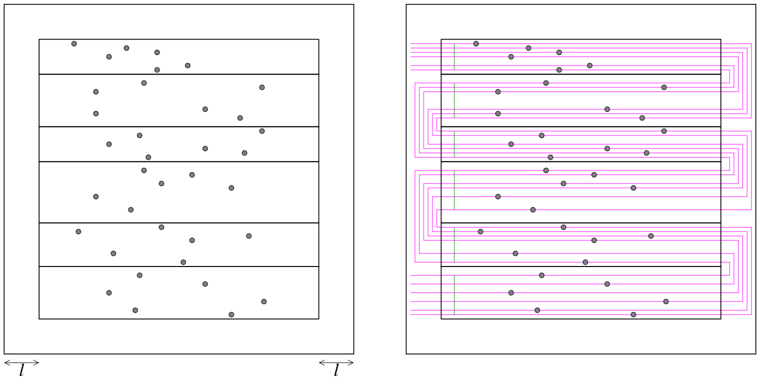

Fig. 3.1. *Left: The grid H, the points in S'', and their grouping. Here $\ell = 6$. Right: Construction of the minor $\ell \times \ell$ grid R passing through the points in S''.*

smaller set S'', because $|S'| = s_y \geq \sqrt{s} \geq (\lfloor \sqrt[4]{s} \rfloor)^2 = \ell^2$. We divide these ℓ^2 vertices into ℓ groups each of exactly ℓ consecutive vertices according to the order of their y coordinates. Now we have the situation shown on the left of Figure 3.1.

We construct the minor grid R as shown on the right of Figure 3.1. Because each y coordinate is unique, we can draw long horizontal segments through every point. The ℓ columns on the left-hand and right-hand sides of H allow us to connect these horizontal segments together into ℓ vertex-disjoint paths, each passing through exactly ℓ vertices of S''. These paths can be connected by vertical segments within each group. By contracting each horizontal segment into a single vertex, and some further contraction, we can obtain the desired $\ell \times \ell$ grid R as a minor. Each vertex of this grid R is a contraction of at least one vertex in S'' (and hence in S) and other vertices in H. ▯

Lemma 3.2. *Let $G \in \mathcal{L}(f)$ be a graph containing the $m \times m$ grid H as a subgraph, $m > 2\ell$, where $\ell = f(2) + 1$. Then the central $(m - 2\ell) \times (m - 2\ell)$ subgrid H' has the property that every vertex $v \in V(G)$ is adjacent to less than ℓ^4 vertices in H'.*

Proof. Suppose for contradiction that there is a vertex $v \in V(G)$ such that $S = N_G[v] \cap V(H)$ has size $s = |S| \geq \ell^4$. By Lemma 3.1, H' has as a minor a $\ell \times \ell$ grid R such that each vertex in R is a contraction of at least one vertex in S and other vertices in H'. If we perform these contractions and deletions in G, then v is adjacent to all vertices in R. Define $R + v$ to be the grid R plus the vertex v (if v is not already in R) and the star of connections between v and all vertices in R. Then $R + v$ is a minor of G, but has diameter 2 and treewidth $\ell \geq f(2) + 1$, a contradiction. ▯

4. Main theorem.

Now we are ready to present the main result of this paper.

Theorem 4.1. *Let P be a contraction-bidimensional parameter. A minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P if $\mathcal{F}$ is of bounded local treewidth. In particular, for any g-contraction-bidimensional parameter P, function $f\colon \mathbb{N} \to \mathbb{N}$ and any graph $G \in \mathcal{L}(f)$ on which P has value at most k, we have $\mathbf{tw}(G) \leq 2^{O(g^{-1}(k) \log g^{-1}(k))}$. (The constant in the O notation depends on $f(1)$ and $f(2)$.)*

Proof. Let $r = f(1) + 1$ and $\ell = f(2) + 1$. Let $G \in \mathcal{L}(f)$ be a graph on which the graph parameter P has value k. Let m be the largest integer such that $\mathbf{tw}(G) \geq m^{4r^2(m+2)}$. Without loss of generality, we assume G is connected, and

$m > 2\ell$ (otherwise, $\mathbf{tw}(G)$ is a constant because both r and ℓ are constants.) Then G has no complete graph K_r as a minor. By Theorem 2.1, G contains an $m \times m$ grid H as a minor. Thus there exists a sequence of edge contractions and edge/vertex deletions reducing G to H. We apply to G the edge contractions from this sequence, we ignore the edge deletions, and instead of deletion of a vertex v, we only contract v into one of its neighbors. Call the new graph G', which has the $m \times m$ grid H as a subgraph and in addition $V(G') = V(H)$. Because graph parameter P is contraction-bidimensional, its value on G' will not increase. By Lemma 3.2, we know that the central $(m - 2\ell) \times (m - 2\ell)$ subgrid H' of H has the property that every vertex $v \in V(G')$ is adjacent to less than ℓ^4 vertices in H'.

Now, suppose in graph G', we further contract all 2ℓ boundary rows and 2ℓ boundary columns into two boundary rows and two boundary columns (one on each side) and call the new graph G''. Note that here G'' and H' have the same set of vertices. The degree of each vertex of G'' to the vertices that are not on the boundary is at most $(\ell + 1)^2\ell^4$, which is a constant because ℓ is a constant. Here the factor $(\ell+1)^2$ is for the boundary vertices each of which is obtained by contraction of at most $(\ell + 1)^2$ vertices. Again because graph parameter P is contraction-bidimensional, its value on G'' does not increase and thus it is at most p. On the other hand, because the graph parameter is g-contraction-bidimensional, its value on graph G'' is at least $g(m-2\ell)$. Thus $g^{-1}(k) \geq m-2\ell$, so $m = O(g^{-1}(k))$. By Theorem 2.3, the treewidth of the original graph G is at most $2^{O(g^{-1}(k) \log g^{-1}(k))}$ as desired. ▯

The *apex graphs* A_i, $i = 1, 2, 3, \ldots$, are obtained from the $i \times i$ grid by adding a vertex v adjacent to all vertices of the grid. It is interesting to see that, for a wide range of graph parameters, the inverse of Theorem 4.1 also holds.

LEMMA 4.2. *Let P be any contraction-bidimensional parameter where $P(A_i) = O(1)$ for any $i \geq 1$. A minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P only if $\mathcal{F}$ is of bounded local treewidth.*

Proof. The proof follows from Theorem 2.4. The apex graph A_i, has diameter ≤ 2 and treewidth $\geq i$. So a minor-closed family of graphs with the parameter-treewidth property for P cannot contain all apex graphs and hence it is of bounded local treewidth. ▯

Typical examples of graph parameters satisfying Theorem 4.1 and Lemma 4.2 are dominating set and its generalization q-dominating set, for a fixed constant q (in which each vertex can dominate its q-neighborhood). These graph parameters are $\Theta(r^2)$-contraction-bidimensional and their value is 1 for any apex graph $A_i, i \geq 1$.

We can strengthen the "if and only if" result provided by Theorem 4.1 and Lemma 4.2 with the following lemma. We just need to use the fact that if the value of P is less than the value of P' then the parameter-treewidth property for P implies the parameter-treewidth property for P' as well.

LEMMA 4.3. *Let P be a graph parameter whose value is lower bounded by some contraction-bidimensional parameter and let $P(A_i) = O(1)$ for any $i \geq 1$. Then a minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P if and only if $\mathcal{F}$ is of bounded local treewidth.*

Proof. The "only if" direction is the same as in Lemma 4.2. Suppose now that P' is a contraction-bidimensional parameter where, for any graph G, $P'(G) \leq P(G)$. Applying Theorem 4.1 to P' we obtain that, if $\mathcal{F}$ is of bounded local treewidth, then $\mathcal{F}$ has the parameter-treewidth property for P', which means that there exists a strictly increasing function t such that, for any graph $G \in \mathcal{F}$, $\mathbf{tw}(G) \leq t(P'(G))$. As $P'(G) \leq P(G)$, we have that $\mathbf{tw}(G) \leq t(P(G))$ and thus $\mathcal{F}$ has the parameter-treewidth property for P. □

Lemma 4.3 can be used not only for contraction-bidimensional graph parameters. As an example we mention the *clique-transversal number* of a graph, i.e., the minimum number of vertices meeting all the maximal cliques of a graph. (The clique-transversal number is not contraction-bidimensional because an edge contraction may create a new maximal clique and the value of the clique-transversal number may increase.) It is easy to see that this graph parameter always exceeds the domination number (the size of a minimum dominating set) and that any graph in A_i has a clique-transversal set of size 1.

Another application is the Π-*domination number*, i.e., the minimum cardinality of a vertex set that is a dominating set of G and satisfies some property Π in G. If this property is satisfied for any one-element subset of $V(G)$ then we call it *regular*. Examples of known variants of the parameterized dominating-set problem corresponding to the Π-domination number for some regular property Π are the following parameterized problems: the independent dominating set problem, the total dominating set problem, the perfect dominating set problem, and the perfect independent dominating set problem (see the exact definitions in [1]).

We summarize the previous observations with the following:

COROLLARY 4.4. *Let P be any of the following graph parameters: the minimum cardinality of a dominating set, the minimum cardinality of a q-dominating set (for any fixed q), the minimum cardinality of a clique-transversal set, or the minimum cardinality of a dominating set with some regular property* Π. *A minor-closed family of graphs $\mathcal{F}$ has the parameter-treewidth property for P if and only if $\mathcal{F}$ is of bounded local treewidth. The function $t(k)$ in the parameter-treewidth property is* $2^{O(\sqrt{k}\log k)}$.

5. Algorithmic consequences and concluding remarks.

Courcelle [10] proved a meta-theorem on graphs of bounded treewidth; he showed that, if ϕ is a property of graphs that is definable in monadic second-order logic, then ϕ can be decided in linear time on graphs of bounded treewidth. Frick and Grohe [27] partially extended this result to graphs of bounded local treewidth; they showed that, for any property ϕ that is definable in first-order logic and for any class of graphs of bounded local treewidth, there is an $O(n^{1+\varepsilon})$-time algorithm deciding whether a given graph has property ϕ, for any $\varepsilon > 0$. The constant in the O notation depends on $1/\varepsilon$, ϕ, and the local treewidth bound. However, the running time of Frick and Grohe's algorithm remains unanalyzed in terms of ϕ: their algorithm transforms ϕ into so-called "Gaifman normal form" [28] and the complexity of this transformation is unknown.

Using Theorems 2.3 and 4.1, we obtain a result along similar lines of Frick and Grohe. Specifically, consider any property that is solvable in polynomial time on graphs of bounded treewidth, e.g., properties definable in monadic second-order logic. If the property is minor-bidimensional, we obtain a fixed-parameter algorithm on general minor-closed graph families excluding some fixed graphs; and if the property is contraction-bidimensional, we obtain a fixed-parameter algorithm on minor-closed graph families of bounded local treewidth. The differences between our result and Frick and Grohe's result are that our properties must be bidimensional but need not be definable in first-order logic, and our graph families must be minor-closed but need not have bounded local treewidth (for minor-bidimensional properties). Also, in contrast to the work of Frick and Grohe, the running time of our algorithm has an explicit bound.

THEOREM 5.1. *Let P be a graph parameter such that, given a tree decomposition of width at most w for a graph G, the graph parameter can be computed in* $h(w)n^{O(1)}$

time. Now, if P is a g-minor-bidimensional parameter and G belongs to a minor-closed graph family excluding some fixed graphs, or P is a g-contraction-bidimensional parameter and G belongs to a minor-closed family of graphs of bounded local treewidth, then we can compute P on G in $h(2^{O(g^{-1}(k)\log g^{-1}(k))})n^{O(1)} + 2^{2^{O(g^{-1}(k)\log g^{-1}(k))}}n^{3+\varepsilon}$ time, for any $\varepsilon > 0$.

Proof. The algorithm is as follows. First we check whether $\mathbf{tw}(G)$ is in $2^{O(g^{-1}(k)\log g^{-1}(k))}$. By Theorems 2.3 and 4.1, if it is not, graph parameter P has value more than k on graph G. This step can be performed by Amir's algorithm [5], which for a given graph G and integer ω, either reports that the treewidth of G is at least ω, or produces a tree decomposition of width at most $(3+\frac{2}{3})\omega$ in time $O(2^{3.698\omega}n^3\omega^3\log^4 n)$. Thus by using Amir's algorithm we can either compute a tree decomposition of G of size $2^{O(g^{-1}(k)\log g^{-1}(k))}$ in time $2^{2^{O(g^{-1}(k)\log g^{-1}(k))}}n^{3+\varepsilon}$, or conclude that the treewidth of G is not in $2^{O(g^{-1}(k)\log g^{-1}(k))}$.

Now if we find a tree decomposition of the aforementioned width, we can compute P on G in time $h(2^{O(g^{-1}(k)\log g^{-1}(k))})n^{O(1)}$ time. The running time of this algorithm is the one mentioned in the statement of the theorem. □

For example, let G be a graph from a minor-closed family $\mathcal{F}$ of bounded local treewidth. Because the dominating set of a graph with a given tree decomposition of width at most ω can be computed in time $O(2^{2\omega}n)$ [1], Theorem 5.1 gives an algorithm which either computes a dominating set of size at most k, or concludes that there is no such a dominating set in $2^{2^{O(\sqrt{k}\log k)}}n^{O(1)}$ time. The same result holds also for computing the minimum size of a q-dominating set. Indeed, Theorem 5.1 can be applied because the q-dominating set of a graph with a given tree decomposition of width at most ω can be computed in time $O(q^{O(\omega)}n)$ [12]. Also, algorithms on graphs of bounded treewidth for clique-transversal set, and Π-domination set appeared in [8] and [1] respectively. Using these algorithms, and the fact that all these graph parameters are lower bounded by the domination number, the methodology of the proof of Theorem 5.1 can give algorithmic results for clique-transversal set and Π-domination set with the same running times as in the case of dominating set (i.e., $2^{2^{O(\sqrt{k}\log k)}}n^{O(1)}$).

Clearly, the algorithmic results of this paper are mainly of theoretical importance. Towards more practical algorithms, we mention some open problems. It is known that, for any planar graph G with dominating set of size at most k, we have $\mathbf{tw}(G) = O(\sqrt{k})$. The same holds for many other graph parameters [1]. The same bound has also been proved for more general graph classes like graphs of bounded genus [13, 26, 16] and minor-closed graph families of bounded local treewidth [14]. It is natural to ask whether such a smaller bound holds in the case of any bidimensional parameter. This would provide subexponential fixed-parameter algorithms on minor-closed graph families of bounded local treewidth for any such graph parameter.

It is known that the dominating set problem admits a linear size kernel on planar graphs [3]. Recently, this result was extended to graphs of bounded genus [26]. It is tempting to ask whether such a kernel exists for any minor-closed graph class of bounded local treewidth, i.e., any minor-closed graph class with the parameter-treewidth property for dominating set. The same question can be asked for other bidimensional parameters. In particular, we wonder whether there is any link between linear kernels and bidimensionality.

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REFERENCES

[1] J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier, *Fixed parameter algorithms for dominating set and related problems on planar graphs*, Algorithmica, 33 (2002), pp. 461–493.

[2] J. Alber, H. Fan, M. R. Fellows, H. Fernau, R. Niedermeier, F. A. Rosamond, and U. Stege, *Refined search tree technique for dominating set on planar graphs*, Proceedings of the 26th International Symposium Mathematical Foundations of Computer Science (MFCS 2001), Springer LNCS vol. 2136, 2001, pp. 111–122.

[3] J. Alber, M. R. Fellows, and R. Niedermeier, *Efficient data reduction for dominating set: A linear problem kernel for the planar case*, Proceedings of the 8th Scandinavian Workshop on Algorithm Theory (SWAT 2002), Springer LNCS vol. 2368, Berlin, 2002, pp. 150–159. To appear in the Journal of the ACM.

[4] J. Alber, H. Fernau, and R. Niedermeier, *Parameterized complexity: Exponential speed-up for planar graph problems*, accepted for publication in Journal of Algorithms (2004).

[5] E. Amir, *Efficient approximation for triangulation of minimum treewidth*, Proceedings of Uncertainty in Artificial Intelligence: Proceedings of the Seventeenth Conference (UAI-2001), Morgan Kaufmann Publishers, 2001, pp. 7–15.

[6] B. S. Baker, *Approximation algorithms for NP-complete problems on planar graphs*, Journal of the ACM, 41 (1994), pp. 153–180.

[7] H. L. Bodlaender, *A tourist guide through treewidth*, Acta Cybernetica, 11 (1993), pp. 1–23.

[8] M. S. Chang, T. Kloks, and C. M. Lee, Maximum clique transversals, *Proceedings of the 27th International Workshop on Graph-Theoretic Concepts in Computer Science*, Springer LNCS vol. 2204, 2001, pp. 300–310.

[9] J. Chen, I. A. Kanj, and W. Jia. *Vertex cover: further observations and further improvements*, *Journal of Algorithms*, 41(2) (2001), pp. 280–301.

[10] B. Courcelle, *Graph rewriting: an algebraic and logic approach*, In Handbook of theoretical computer science, Vol. B, pp. 193–242. Elsevier, Amsterdam, 1990.

[11] E. D. Demaine, F. V. Fomin, M. T. Hajiaghayi, and D. M. Thilikos, *Bidimensional Parameters and Local Treewidth*, Proceedings of the 6th Latin American Theoretical Informatics Symposium (LATIN 2004), Springer LNCS vol. 2976, 2004, pp. 109–118.

[12] E. D. Demaine, F. V. Fomin, M. T. Hajiaghayi, and D. M. Thilikos, *Fixed-Parameter Algorithms for the (k, r)-Center in Planar Graphs and Map Graphs*, Proceedings of the 30th International Colloquium on Automata, Languages and Programming (ICALP 2003), Springer, LNCS vol. 2719, 2003, pp. 829-844.

[13] E. D. Demaine, F. V. Fomin, M. T. Hajiaghayi, and D. M. Thilikos, *Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs*, Proceedings of the 15th ACM-SIAM Symposium on Discrete Algorithms (SODA 2004), pp. 823–832.

[14] E. D. Demaine and M. T. Hajiaghayi *Equivalence of Local Treewidth and Linear Local Treewidth and its Algorithmic Applications*, Proceedings of the 15th ACM-SIAM Symposium on Discrete Algorithms (SODA 2004), pp. 833–842.

[15] E. D. Demaine, M. T. Hajiaghayi, and D. M. Thilikos, *Exponential speedup of fixed parameter algorithms on $K_{3,3}$-minor-free or K_5-minor-free graphs*, Proceedings of the 13th International Symposium on Algorithms and Computation (ISAAC 2002), Springer LNCS vol. 2518, 2002, pp. 262–273.

[16] E. D. Demaine, M. T. Hajiaghayi, and D. M. Thilikos, *The Bidimensional Theory of Bounded-Genus Graphs*, manuscript.

[17] R. Diestel, *Graph theory*, Springer-Verlag, New York, 2000.

[18] R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen, *Highly connected sets and the excluded grid theorem*, Journal of Combinatorial Theory Series B, 75 (1999), pp. 61–73.

[19] R. G. Downey and M. R. Fellows, *Parameterized complexity*, Springer-Verlag, New York, 1999.

[20] J. Ellis, H. Fan, and M. Fellows, *The dominating set problem is fixed parameter tractable for graphs of bounded genus*, Proceedings of the 8th Scandinavian Workshop on Algorithm Theory (SWAT 2002), Springer LNCS vol. 2368, 2002, pp. 180–189.

[21] D. Eppstein, *Diameter and treewidth in minor-closed graph families*, Algorithmica, 27 (2000), pp. 275–291.

[22] J. Flum and M. Grohe, *Fixed-parameter tractability, definability, and model-checking*, SIAM Journal on Computing 31 (2001) pp. 113–145.

[23] F. V Fomin and D. M. Thilikos, *Dominating sets in planar graphs: branch-width and exponential speed-up*, Proceedings of the Fourteenth ACM-SIAM Symposium on Discrete Algorithms (SODA 2003), pp. 168–177.

[24] F. V. Fomin and D. M. Thilikos, *Dominating sets and local treewidth*, Proceedings of the 11th European Symposium on Algorithms (ESA 2003), Springer LNCS vol. 2832, 2003, pp. 221-229.

[25] F. V. Fomin and D. M. Thilikos, *A Simple and Fast Approach for Solving Problems on Planar Graphs*, Proceedings of the 21st International Symposium on Theoretical Aspects of Computer Science (STACS 2004), Springer LNCS vol. 2996, 2004, pp. 56–67.

[26] F. V. Fomin and D. M. Thilikos, *Parameterized algorithms for Dominating Set in graphs on surfaces: Linear kernel and exponential speed-up*, Proceedings of the 31th International Colloquium on Automata, Languages and Programming (ICALP 2004), Springer, LNCS, to appear.

[27] M. Frick and M. Grohe, *Deciding first-order properties of locally tree-decomposable graphs*, Journal of the ACM, 48 (2001), pp. 1184 – 1206.

[28] H. Gaifman, *On local and nonlocal properties*, Proceedings of the Herbrand Symposium (Marseilles, 1981), pages 105–135, North-Holland, Amsterdam, 1982.

[29] M. Grohe, *Local tree-width, excluded minors, and approximation algorithms*, Combinatorica 23(4) (2003), pp. 613–632.

[30] I. Kanj and L. Perković, *Improved parameterized algorithms for planar dominating set*, Proceedings of the 27th International Symposium Mathematical Foundations of Computer Science (MFCS 2002), Springer LNCS vol. 2420, 2002, pp. 399–410.

[31] N. Robertson and P. D. Seymour, *Graph minors. II. Algorithmic aspects of tree-width*, Journal of Algorithms 7 (1986), pp. 309–322.

[32] N. Robertson and P. D. Seymour, *Graph minors. V. Excluding a planar graph*, Journal of Combinatorial Theory Series B, 41 (1986), pp. 92–114.