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Continuous Blooming of Convex Polyhedra

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Abstract We construct the first two continuous bloomings of all convex polyhedra. First, the source unfolding can be continuously bloomed. Second, any unfolding of a convex polyhedron can be refined (further cut, by a linear number of cuts) to have a continuous blooming.

1 Introduction

A standard approach to building 3D surfaces from rigid sheet material, such as sheet metal or cardboard, is to design an *unfolding*: cuts on the 3D surface so that the remainder can unfold (along edge hinges) into a single flat non-self-overlapping piece. The advantage of this approach is that many (relatively cheap) technologies—such as NC machining/milling, laser cutters, waterjet cutters, and sign cutters—enable manufacture of an arbitrary flat shape (with hinges) from a sheet of material. As a result, existence of and algorithms for unfolding have been studied extensively in the mathematical literature.

Often overlooked in this literature, however, is the second manufacturing step: can we actually fold the flat shape along the hinges into the desired 3D surface, without

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self-intersection throughout the motion? Some unfoldings have no such motion [BLS05, Theorem 4]. In this paper, we develop such motions for large families of unfoldings of convex polyhedra. In particular, we establish for the first time that every convex polyhedron can be manufactured by folding an unfolding.

Unfolding. More precisely, an *unfolding* of a polyhedral surface in 3D consists of a set of cuts (arcs) on the surface whose removal result in a surface whose intrinsic metric is isometric to the interior of a flat non-self-overlapping polygon, called the *development* or *unfolded shape* of the unfolding. We demand that no two interior points of the polygon coincide, but the boundary of the polygon may self-touch. The cuts in an unfolding of a convex polyhedron form a tree, necessarily spanning all vertices of the polyhedron (enabling them to flatten) [BDE+03]. All unfoldings we consider allow cuts anywhere on the polyhedral surface, not just along polyhedron edges. Four general unfolding algorithms are known for arbitrary convex polyhedra: the source unfolding [SS86, MP08], the star unfolding [AO92], and two variations thereof [IOV10, IOV09]. Positive and negative results for unfolding nonconvex polyhedra can be found in [BDE+03, DFO07, O'R08].

Blooming. Imagine the faces of the cut surface in an unfolding as rigid plates, and the (sub)edges of the polyhedron connecting them as hinges. A *continuous blooming* of an unfolding is a continuous motion of this plate-and-hinge structure from the original shape of the 3D polyhedron (minus the cuts) to the flat shape of the development, while avoiding intersection between the plates throughout the motion. In 1999, Biedl, Lubiw, and Sun [BLS05] gave an example of an unfolding of an orthogonal (nonconvex) polyhedron that cannot be continuously bloomed. In 2003, Miller and Pak [MP08, Conjecture 9.12] reported the conjecture of Connelly that every convex polyhedron has a nonoverlapping unfolding that can be continuously bloomed. He further conjectured that the blooming can monotonically open all dihedral angles of the hinges. More recently, Pak and Pinchasi [PP09] describe a simple blooming algorithm for convex polyhedra, which they show works for the finite class of Archimedean solids (excluding prisms and antiprisms, by extending existing bloomings of Platonic solids) but fails on other polyhedra. Aloupis et al. [ADL+08] analyze the unfoldings of polyhedral bands, formed by the intersection of the surface of a convex polyhedron with the space between two parallel planes, as long as this space does not contain any vertices of the polyhedron. For the case where the orthogonal projection of the band boundaries are nested, they show that the band can be continuously bloomed. The result was later generalized to the case of arbitrary bands by Aloupis [Alo05]. No other nontrivial positive results have been established.

Our results. We prove Connelly's conjecture by giving the first two general algorithms for continuous blooming of certain unfoldings of arbitrary convex polyhedra, which also monotonically open all hinge dihedral angles. Both of our algorithms have a relatively simple structure: they perform a sequence of linearly many steps of the form "open one dihedral angle uniformly by angle α". Thus our algorithms open angles monotonically, uniformly (at constant speed), and one at a time. The challenge in each case is to prove that the motions cause no intersection.

First we show in Section 2 that every unfolding can be refined (further cut, by a linear number of cuts) into another unfolding with a continuous blooming. Indeed, we show that any serpentine unfolding (whose dual tree is a path) has a continuous blooming, and then standard techniques can refine any unfolding into a serpentine unfolding.

Next we show in Section 3 that one particularly natural unfolding, the source unfolding, has a continuous blooming. It is in the context of the source unfolding that Miller and Pak [MP08] described continuous blooming.

Finally we mention in Section 4 an unsolved more-general form of continuous blooming for the source unfolding that resembles Cauchy's arm lemma.

2 Blooming a Refinement of Any Unfolding

In this section, we show that any unfolding can be refined into an unfolding with a continuous blooming. An unfolding U is a *refinement* of another unfolding U' if the cuts in U form a superset of the cuts in U'. Our approach is to make the unfolding serpentine: an unfolding is *serpentine* if the dual tree of the faces is a path. Here "faces" is a combinatorial notion, not necessarily coinciding with geometry: we allow artificial edges with flat dihedral angles. We do require, however, that every two faces adjacent in the dual path form a nonflat dihedral angle; otherwise, those faces could be merged. We use the term *facet* when referring to the original geometric facets of the polyhedron Q.

Lemma 1 *Every unfolding can be refined by a linear number of additional cuts into a serpentine unfolding.*

Proof: The polyhedron and unfolding together define *faces* bounded by polyhedron edges and by cuts in the unfolding. For every two such faces that share an uncut edge, we refine by adding a cut from the centroid of each face to the midpoint of the shared edge. Figure 1 shows an example. These cuts form a tree isomorphic to the dual tree of the unfolding, with a node for each face and an arc connecting two faces that share an uncut edge in the original unfolding. Thus, after the refinement, the remaining subfaces can be connected together in a Hamiltonian cycle via an Euler tour around the tree of added cuts. (This powerful trick originates in [AHMS96, Theorem 2.4] for the case of triangulations.) Finally, we make one additional cut along a (sub)edge to divide the dual cycle into a dual path. If the original polyhedron and unfolding define f faces, the total number of added cuts is $2(f-1)+1 = 2f-1$.

We observe that any refinement that preserves connectivity (such as this one) also preserves that the cuts form a valid unfolding, with a non-self-overlapping development. The additional cuts in the refinement are intrinsic to the surface metric, so they can be applied to the unfolding just as well both on the 3D polyhedron and developed in the plane. Hence the refined unfolding develops in the plane to a region with exactly the same closure as the original unfolding, implying that the refined unfolding has no (interior) self-overlap if and only if the original unfolding has none. □

It remains to prove that such a refinement suffices:

Theorem 1 *Any serpentine unfolding can be continuously bloomed.*

Corollary 1 *Any unfolding of a polyhedron can be refined by a linear number of additional cuts into an unfolding with a continuous blooming.*

Toward Theorem 1, we start with the following simple blooming algorithm. Figure 2 shows an example of the algorithm in action.

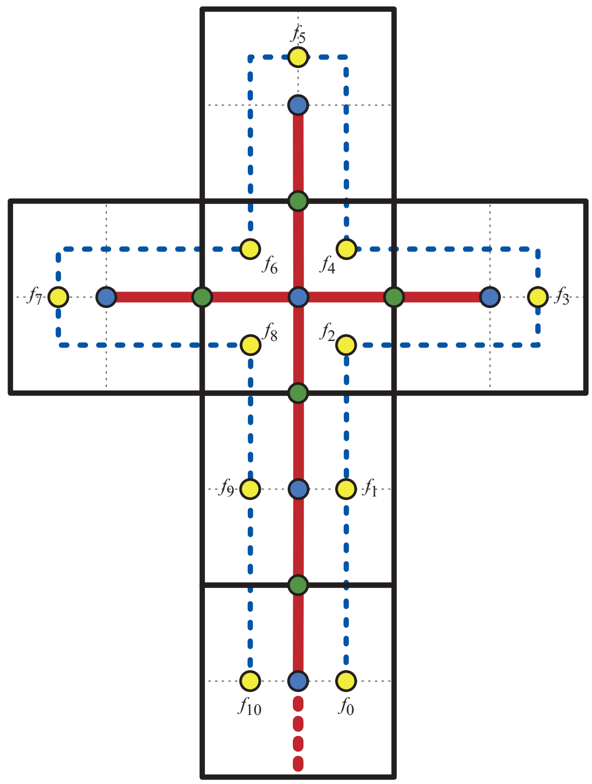

Fig. 1 Refinement of the Latin cross unfolding of a cube to a serpentine unfolding. The dashed cut at the base of the cross is the one additional cut that divides the cycle to a path.

Algorithm 1 (Path-Unroll) Suppose we are given a serpentine unfolding of a polyhedron Q whose dual path is $P = \langle f_0, f_1, \ldots, f_k \rangle$. For $i = 1, 2, \ldots, k$ in sequence, uniformly open the dihedral angle between f_{i-1} and f_i until those two faces become coplanar.

The Path-Unroll algorithm is almost what we need: we show below that it only causes touching, not crossing, between faces that are coplanar with a facet of Q and touching between faces whose supporting planes intersect in a line supporting an edge of Q. Precisely, two (convex) sets are *noncrossing* if there is a plane H such that both sets lie in opposite closed halfspaces bounded by H, while two sets *touch* if they have a common intersection but are noncrossing.

Lemma 2 *The Path-Unroll algorithm causes no crossing between faces of the unfolding. Furthermore, (two-dimensional) touching between faces can occur only at the end of step i between faces coplanar with the facet containing f_i, and between the beginning and the end of step i at the edge of Q shared by f_{i-1} and f_i.*

Proof: The Path-Unroll algorithm has the invariant that, at step i of the algorithm, the prefix $f_0, f_1, \ldots, f_{i-1}$ of faces lies entirely in the plane containing f_{i-1}, and because this planar prefix appears as a subset of the final (nonoverlapping) unfolding, it does not

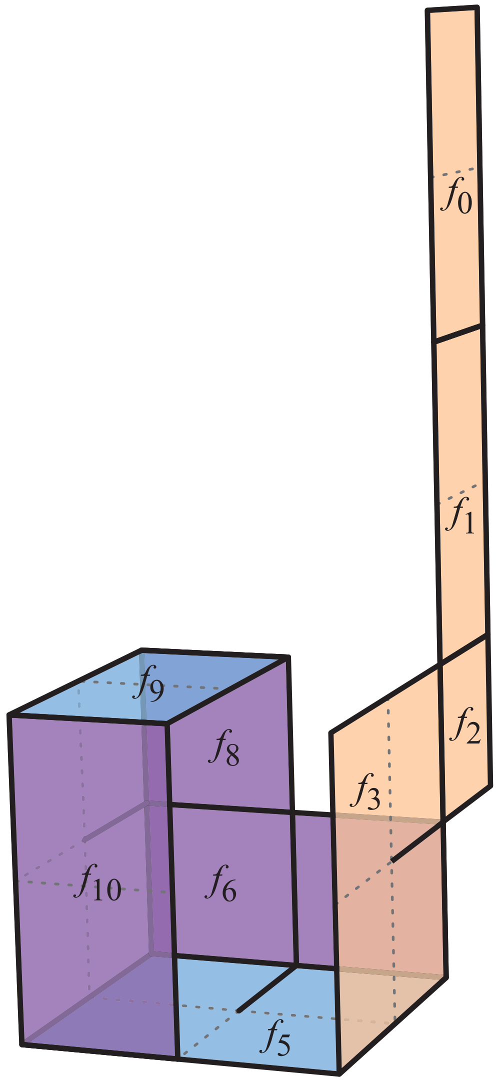

Fig. 2 The Path-Unroll algorithm applied to the refined cube unfolding of Figure 1, after the first $i = 3$ steps.

self-intersect. The suffix $f_i, f_{i+1}, \ldots, f_k$ of faces is a subset of the original polyhedral surface, so does not self-intersect.

It remains to show that the prefix and suffix do not cross each other and intersections occur only as described in the statement of the lemma. At the beginning of step i of the path unroll algorithm, the plane containing f_{i-1} intersects the polyhedron at precisely the facet of Q containing f_{i-1}. At the end of step i, the plane intersects the polyhedron at precisely the facet of Q containing f_i. In the middle of step i, the plane intersects the polyhedron at precisely the edge of Q bounding f_{i-1} and f_i; at all times, the plane remains tangent to the polyhedron. Thus the prefix $f_0, f_1, \ldots, f_{i-1}$

only touches the suffix $f_i, f_{i+1}, \ldots, f_k$ at a facet of Q at the beginning and end of step i, and at an edge of Q during step i. □

To avoid touching at faces, we modify the algorithm slightly.

Algorithm 2 (Path-TwoStep) Suppose we are given a serpentine unfolding of a polyhedron Q whose dual path is $P = \langle f_0, f_1, \ldots, f_k \rangle$. Let $0 < \phi_i < \pi$ be the dihedral angle between f_{i-1} and f_i, let $\psi_i = \pi - \phi_i$ be the required unfolding angle, and let $\Psi_i = \sum_{j=1}^{i} \psi_j$ be the prefix sum for $i = 1, 2, \ldots, k$. Choose $\varepsilon > 0$ small enough.

- At time $t \in [0, \Psi_1 - \varepsilon]$, uniformly open the dihedral angle between f_0 and f_1 until they form a dihedral angle of $\pi - \varepsilon$.
- For $i = 2, 3, \ldots, k$ (see the timing diagram in Figure 3):
  - At time $t \in [\Psi_{i-1} - \varepsilon, \Psi_i - 2\varepsilon]$, uniformly open the dihedral angle between f_{i-1} and f_i until they form a dihedral angle of $\pi - \varepsilon$.
  - At time $t \in [\Psi_i - 2\varepsilon, \Psi_i - \varepsilon]$, uniformly open the dihedral angle between f_{i-2} and f_{i-1} until those two faces become coplanar.
- At time $t \in [\Psi_k - \varepsilon, \Psi_k]$, uniformly open the dihedral angle between f_{k-1} and f_k until those two faces become coplanar.

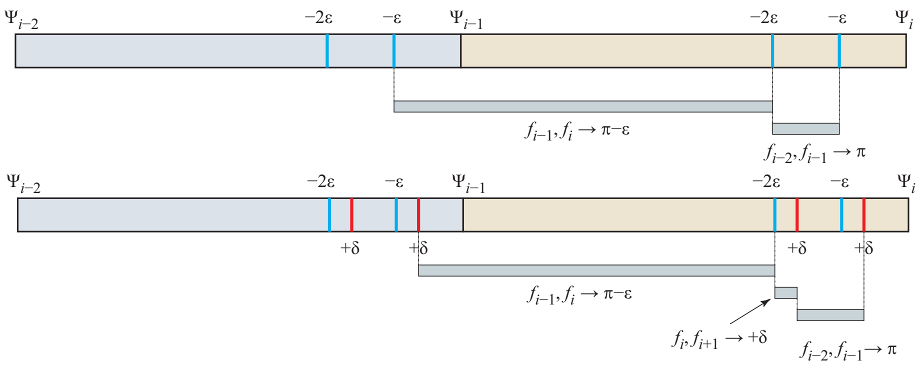

Fig. 3 Timing diagrams for Path-TwoStep (top) and Path-Waltz (bottom).

This algorithm avoids two-dimensional touching but may still cause one-dimensional touching.

Lemma 3 *The Path-TwoStep algorithm causes no crossing between faces of the unfolding. Furthermore, touching can occur only at time $t = \Psi_i - \varepsilon$ for $i = 1, 2, \ldots, k$ at the edge of Q shared by f_{i-1} and f_i.*

Proof: Let ℓ_j be the supporting line of the edge shared by f_{j-1} and f_j. Let q_j be the intersection point (in projective space) between the lines ℓ_j and ℓ_{j-1}, which lie in the common plane of f_{j-1}. At times $t \in [\Psi_{i-1} - \varepsilon, \Psi_i - \varepsilon]$, consider the central projection of Q and the unfolding, from point q_i onto a plane perpendicular to the bisector of ℓ_i and ℓ_{i-1}, as shown in Figure 4. (If q_i is a point at infinity, this projection is an orthogonal projection onto a plane perpendicular to the direction of q_i.) In that projection, Q projects to a convex polygon Q', and so the projection of the suffix $f_i, f_{i+1}, \ldots, f_k$

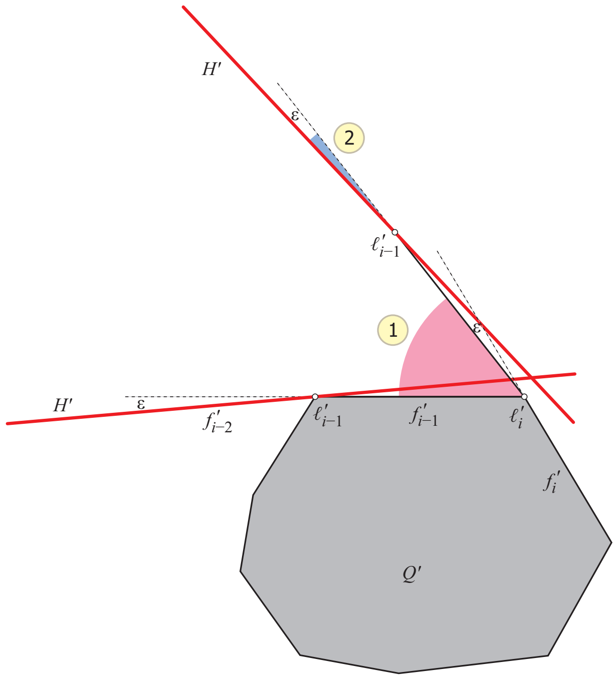

Fig. 4 The central projection from point q_i. During $t \in [\Psi_{i-1} - \varepsilon, \Psi_i - 2\varepsilon]$, rotation (marked 1 in the figure) occurs in projection about ℓ'_i. During $t \in [\Psi_i - 2\varepsilon, \Psi_i - \varepsilon]$, rotation (marked 2) straightens the angle at the new position of ℓ'_{i-1}.

of untouched faces is within that polygon. Face f_{i-1} projects to the segment f'_{i-1} connecting the points ℓ'_i and ℓ'_{i-1} representing the lines ℓ_i and ℓ_{i-1}. Just as in the invariant of Lemma 2, the prefix $f_0, f_1, \ldots, f_{i-2}$ lies in the plane H of f_{i-2}, which is incident to ℓ_{i-1} and thus projects to a line H'.

At time $t = \Psi_{i-1} - \varepsilon$, the face f_{i-1} is in its original position and so does not intersect any of the faces $f_i, f_{i+1}, \ldots, f_k$. The projection of $f_0, f_1, \ldots, f_{i-2}$ is contained in line H', which is tangent to Q' at point ℓ'_{i-1}, assuming $\varepsilon < \psi_{i-1}$. When $t \in [\Psi_{i-1} - \varepsilon, \Psi_i - 2\varepsilon]$, as the angle between f_i and f_{i-1} opens, the line containing edge f'_{i-1} remains tangent to Q' while rotating about point ℓ'_i. Thus f_{i-1} touches Q exactly at its edge connecting it to f_i and avoids any intersection.

During the motion, H' remains tangent to the convex hull of Q' and f'_{i-1} at the projection of ℓ_{i-1}, assuming ε is chosen so that $\pi - \varepsilon$ is larger than the dihedral angle of the edge in ℓ_{i-1} of the convex hull of the suffix $f_{i-1}, f_i, \ldots, f_k$ when the dihedral angle between f_i and f_{i-1} is π.

At times $t \in [\Psi_i - 2\varepsilon, \Psi_i - \varepsilon]$, as the dihedral angle between f_{i-2} and f_{i-1} opens until those two faces become coplanar, H' remains tangent to the convex hull of Q' and f'_{i-1} at the projection of ℓ_{i-1} and so does not intersect Q' until time $\Psi_i - \varepsilon$, when H' becomes collinear with f'_{i-1}. At precisely that instant, H' touches Q' at the projection of ℓ_i, and so the faces in the prefix $f_0, f_1, \ldots, f_{i-1}$ might touch, but not cross, edges still on Q at the edge between f_{i-1} and f_i. □

To avoid this one-dimensional touching, we modify the algorithm once more.

Algorithm 3 (Path-Waltz) Suppose we are given a serpentine unfolding of a polyhedron Q whose dual path is $P = \langle f_0, f_1, \ldots, f_k \rangle$. Let $0 < \phi_i < \pi$ be the dihedral angle between f_{i-1} and f_i, let $\psi_i = \pi - \phi_i$ be the required folding angle, and let $\Psi_i = \sum_{j=1}^{i} \psi_j$ be the prefix sum for $i = 1, 2, \ldots, k$. Choose $\varepsilon > \delta > 0$ small enough.

- At time $t \in [0, \Psi_1 - \varepsilon]$, uniformly open the dihedral angle between f_0 and f_1 until they form a dihedral angle of $\pi - \varepsilon$.
- At time $t \in [\Psi_1 - \varepsilon, \Psi_1 - \varepsilon + \delta]$, uniformly open the dihedral angle between f_1 and f_2 by an angle δ.
- For $i = 2, 3, \ldots, k-1$:
  - At time $t \in [\Psi_{i-1} - \varepsilon + \delta, \Psi_i - 2\varepsilon]$, uniformly open the dihedral angle between f_{i-1} and f_i until they form a dihedral angle of $\pi - \varepsilon$.
  - At time $t \in [\Psi_i - 2\varepsilon, \Psi_i - 2\varepsilon + \delta]$, uniformly open the dihedral angle between f_i and f_{i+1} by an angle δ.
  - At time $t \in [\Psi_i - 2\varepsilon + \delta, \Psi_i - \varepsilon + \delta]$, uniformly open the dihedral angle between f_{i-2} and f_{i-1} until those two faces become coplanar.
- At time $t \in [\Psi_{k-1} - \varepsilon + \delta, \Psi_k - \varepsilon]$, uniformly open the dihedral angle between f_{k-1} and f_k until those two faces become coplanar.
- At time $t \in [\Psi_k - \varepsilon, \Psi_k]$, uniformly open the dihedral angle between f_{k-2} and f_{k-1} until those two faces become coplanar.

Figure 3 shows the timing diagram of the Path-Waltz algorithm. Path-Waltz avoids all touching except that present in the original unfolding, proving our desired theorem about serpentine unfoldings having continuous bloomings:

Proof of Theorem 1: We prove that Path-Waltz causes no intersection. Note that during the times $t \in [\Psi_{i-1} - \varepsilon + \delta, \Psi_i - 2\varepsilon]$ for some i, the configuration is exactly the same as in the Path-TwoStep algorithm, and so no intersection occurs during those times.

Let H_i be the plane containing face f_i. At time $t = \Psi_i - 2\varepsilon$, the configuration occurs in Path-TwoStep, and thus does not self-intersect. Furthermore, the proof of Lemma 3 shows that the plane H_{i-2} does not intersect Q unless the edge shared by f_i and f_{i-1} has a common endpoint with the edge shared by f_{i-1} and f_{i-2} (and that common endpoint would be q_{i-1}). Assume for now that this is not the case. Because the non-self-intersecting configuration space is open, δ can be chosen small enough that, at times $t \in [\Psi_i - 2\varepsilon, \Psi_i - 2\varepsilon + \delta]$, as the angle between f_i and f_{i+1} opens, no faces intersect, and H_{i-2} does not intersect Q. Note that during those times, the suffix $f_{i-1}, f_i, \ldots, f_k$ is in a configuration identical to that in the Path-Twostep algorithm at times $t' \in [\Psi_i - \varepsilon, \Psi_i - \varepsilon + \delta]$ and again, by the proof of Lemma 3, the plane H_{i-1} does not touch the interior of Q. Therefore, the open double-wedge between H_{i-1} and H_{i-2} remains empty throughout the motion. When, at times $t \in [\Psi_i - 2\varepsilon + \delta, \Psi_i - \varepsilon + \delta)$, the angle between f_{i-2} and f_{i-1} opens, the faces in the suffix $f_{i-2}, f_{i-1}, \ldots, f_k$ remain in that open double-wedge and do not cause any intersection.

It remains to show that no intersections occur during the motion if the edge shared by f_i and f_{i-1} and the edge shared by f_{i-1} and f_{i-2} both have q_{i-1} as a common endpoint. For this case, notice that an intersection can occur only in a small neighborhood of point q_{i-1}. The angle ε can be chosen small enough so that the normal vectors to faces f_i, f_{i-1}, and f_{i-2} all have pairwise angles smaller than $\pi/2$, and so the opening does not cause intersections with the double wedge. □

3 Blooming the Source Unfolding

This section proves that the following algorithm continuously blooms the source unfolding of any convex polyhedron. Figure 5 shows a simple example of the algorithm in action.

Algorithm 4 (Tree-Unroll) Let T be the tree of faces formed by the source unfolding, rooted at the face containing the source s of the unfolding. For each face in the order given by a post-order traversal of T, uniformly open the dihedral angle between the face and its parent in T until the two faces become coplanar.

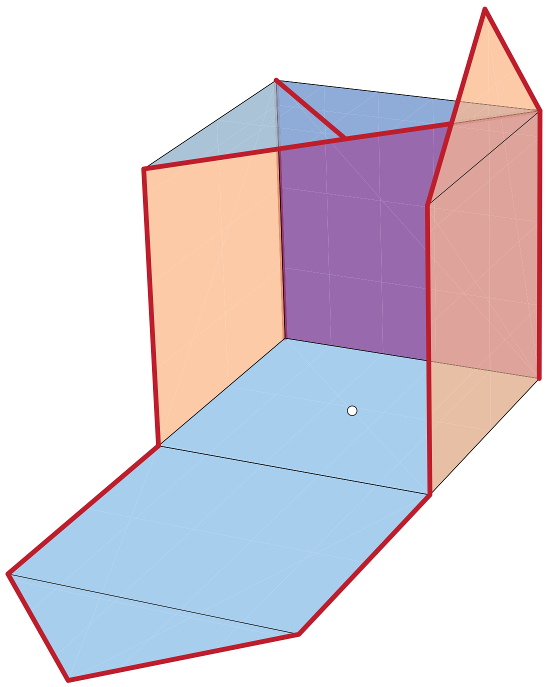

Fig. 5 Start of Algorithm Tree-Unroll applied to the source unfolding of a cube (with cuts drawn in bold lines), with the source point in the center of the bottom face. The front subtree rooted at the bottom face has been unfolded, and the leaf of the right subtree has been unfolded to the plane of its parent face.

First we need some tools for arguing about shortest paths and source unfoldings. Our first tool shows that source unfoldings change little when "growing" a polyhedron:

Lemma 4 *If the interior of a convex polyhedron Q is contained in the interior of a convex polyhedron Q', and p and q are points on the boundary of both Q and Q', then the shortest path between p and q is at least as long on Q' as on Q.*

Proof: The shortest path in 3D between two points p and q on the boundary of a convex polyhedron Q, while avoiding Q as an obstacle, lies entirely on the surface of Q: otherwise, each connected part of the shortest path not intersecting Q would be a segment joining two points on the surface of Q, but then by convexity of Q, that entire segment is in or on Q, a contradiction. Thus the shortest path between p and q on the surface of Q is the same as the shortest 3D path between p and q avoiding obstacle Q. Because the interior of Q is a subset of the interior of Q', the set of 3D paths from p to q avoiding obstacle Q' is a subset of the set of 3D paths from p to q avoiding obstacle Q, so the shortest path in the first set is at least as long as the shortest path in the second set. $\square$

Next we consider the behavior of individual shortest paths during Tree-Unroll, which degenerates to the Path-Unroll algorithm of Section 2:

Lemma 5 *Suppose a shortest path on a convex polyhedron Q is the 3D polygonal path $P = \langle p_0, p_1, \ldots, p_k \rangle$, with segment $p_{i-1}p_i$ contained in face f_i of Q and point p_i on an edge of Q for $0 < i < k$. Consider the motion of the polygonal path P induced by running the Path-Unroll algorithm on the faces $f_1, f_2, \ldots, f_k$. Then the path P does not hit the plane containing f_k (except at $p_{k-1}p_k$) until the very end of the motion, when the whole path P hits that plane.*

Proof: Let h_i be the plane containing face f_i, and let h_i^+ be the closed halfspace bounded by h_i and containing Q. We define the *grown polyhedron* $Q[P] = h_1^+ \cap h_2^+ \cap \cdots \cap h_k^+$ to be the intersection of these halfspaces (which may be unbounded). The interior of $Q[P]$ contains the interior of Q, and the endpoints of P are on the boundary of $Q[P]$, so by Lemma 4, P remains a shortest path in $Q[P]$. The heart of our proof will be maintaining this invariant throughout the motion.

Now consider executing the first step of Path-Unroll, that is, rotating the first face f_1 to open the dihedral angle between f_1 and f_2. As we perform the rotation, we keep the first edge p'_0p_1 of the modified path P' on the rotating face f'_1 and we maintain the grown polyhedron $Q[P']$. Below we prove the following invariant:

Claim 1 *The modified path P' remains a shortest path on the grown polyhedron $Q[P']$, as long as P' remains on the surface of $Q[P']$.*

Assuming this claim for now, one problem can still arise: if p'_0 reaches some plane h_i defining one of the other faces $i > 2$ (and thus enters that face), then we cannot continue rotating f'_1 (leaving f_i behind) while keeping P' on the boundary of $Q[P']$. But if this happens, then P' could not actually be a shortest path, because there would be a shortcut p'_0p_i that lies within h_i, instead of following the path P' between p'_0 and p_i (which is nonplanar because $i > 2$), contradicting the claim that P' is shortest.

Therefore the motion works all the way to the time at which the dihedral angle between f_1 and f_2 becomes flat, reducing the number of faces in $Q[P']$. By induction, we can continue all the way and flatten the entire path, and by the previous argument, only at the end do we hit the final plane h_k. $\square$

It remains to prove Claim 1:

Proof of Claim 1: Suppose for contradiction that there is a shorter path $\tilde{P}$ between p'_0 and p_k on the grown polyhedron $Q[P']$; refer to Figure 6. Let $\tilde{p}$ be the last point along the path $\tilde{P}$ that touches the plane h_1. Such a point exists because p'_0 and p_k are on opposite sides of h_1, given that p_k lies on the polyhedron $Q[P]$ and p'_0 has been rotated from a point p_0 on h_1 by less than $180°$ around a line $h_1 \cap h_2$ on h_1. Because $\tilde{p}$ lies on the plane h_1 and within the halfspaces $h_2^+, h_3^+, \ldots, h_k^+$ (being on the surface of $Q[P']$), $\tilde{p}$ is on the surface of $Q[P]$. Thus the segment $p_0\tilde{p}$ lies on the original face f_1.

Similarly, the subpath $\tilde{P}[\tilde{p}, p_k]$ of $\tilde{P}$ from $\tilde{p}$ to p_k lies on the surface of $Q[P]$, because it lies on $Q[P']$ and remains in the halfspace h_1^+. Because P is a shortest path from p_0 to p_k on Q, its length $|P| = |p_0p_1| + |P[p_1p_k]|$ is at most the length of any particular path, namely, $|p_0\tilde{p}| + |\tilde{P}[\tilde{p}, p_k]|$.

We now show that $|p_0'\tilde{p}| \geq |p_0\tilde{p}|$, which implies that $|\tilde{P}| = |p_0'\tilde{p}| + |\tilde{P}[\tilde{p}, p_k]| \geq |p_0\tilde{p}| + |\tilde{P}[\tilde{p}, p_k]| \geq |P| = |P'|$, contradicting that $\tilde{P}$ was shorter than P'. Consider the triangle $p_0p_1\tilde{p}$, which lies in the plane h_1, and the rotated version of that triangle in h_1', $p_0'p_1\tilde{p}'$. The quadrilateral $\tilde{p}p_0p_0'\tilde{p}'$ is an isosceles trapezoid, because $|p_0\tilde{p}| = |p_0'\tilde{p}'|$ and p_0p_0' is parallel to $\tilde{p}\tilde{p}'$ (both being perpendicular to the bisecting plane of h_1 and h_1'). Therefore the diagonal $\tilde{p}p_0'$ is longer than the equal sides, proving the last claim. □

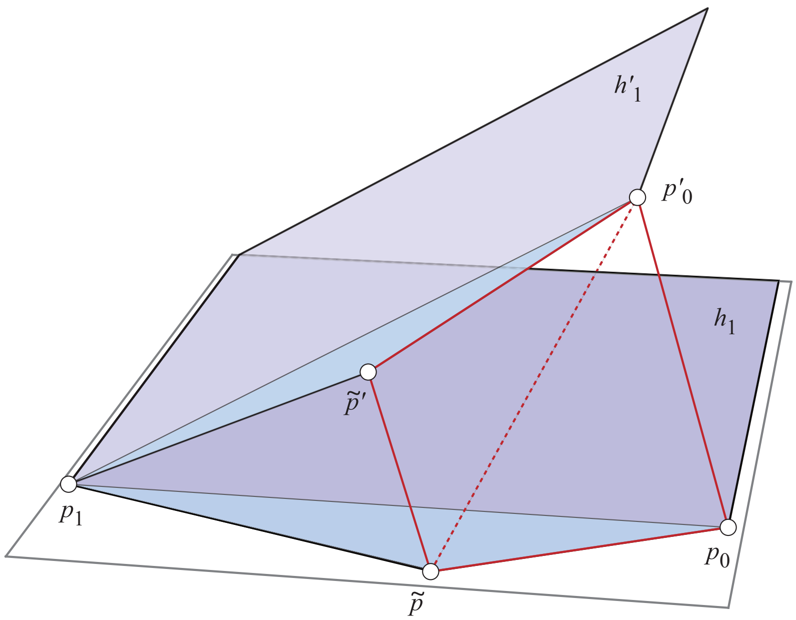

Fig. 6 Proof of Claim 1

Recasting Lemma 5 into the context of Tree-Unroll, we obtain the following:

Lemma 6 *During the recursion of Tree-Unroll on some face f of the tree T, the subtree T_f of faces in T rooted at f does not hit the plane containing the parent $\hat{f}$ of f in T until the algorithm has visited all of T_f, at which time the entire subtree T_f simultaneously touches that plane.*

Proof: Suppose for contradiction that, during the unfolding of T_f, some point p of T_f touches the plane containing $\hat{f}$. By the definition of the source unfolding, the shortest path from the source s to p passes through face $\hat{f}$, say at some point q, and through face f. Then, because Tree-Unroll unfolds that shortest path one face at a time, from p to q, Lemma 5 implies that the entire shortest path from q to p has already been unfolded onto the plane containing $\hat{f}$ (given that p is already there). But then face f has also been unfolded onto the plane containing $\hat{f}$, which happens only when the algorithm has completed the recursion on f. □

Finally we can prove the desired correctness of Tree-Unroll:

Theorem 2 *The Tree-Unroll algorithm causes no intersection during the blooming motion, for any source unfolding of any convex polyhedron.*

Proof: Each step of the algorithm unfolds the faces of the subtree T_f rooted at some face f onto the plane containing the parent $\hat{f}$ of f. During this motion, all other faces are either on the polyhedron Q, or on planes containing the faces of the ancestors of f in T. Because the development of the subtree T_f is a subset of the development of the entire source unfolding, the faces of the subtree T_f do not intersect each other in the plane containing f. Because the faces of T_f are in a supporting plane of the polyhedron Q, those faces do not intersect any of the faces still on Q that are not coplanar with $\hat{f}$. More precisely, the plane containing the subtree T_f intersects Q at the polyhedron face containing f at the beginning of the motion, intersects Q at the edge between f and $\hat{f}$ during the motion, and intersects Q at the polyhedron face containing $\hat{f}$ at the end of the motion. At the end, when T_f and $\hat{f}$ become coplanar, we know from the definition of the source unfolding that there is a shortest path from any point of the developed T_f to a point of $\hat{f}$ that crosses the edge between faces f and $\hat{f}$. Thus, the developed T_f lies on the opposite side of the line containing this edge compared to the entire (convex) polyhedron face containing $\hat{f}$, so $\hat{f}$ and T_f do not intersect in their shared plane. Finally, by Lemma 6, the faces of T_f do not intersect any of the planes containing the faces of the ancestors of f, until the end of the motion when they all hit the plane containing $\hat{f}$, and the previous argument applies. Therefore the blooming motion causes no intersection. □

4 Open Problems

Cauchy's Arm Lemma states that opening the angles of a planar open chain initially in convex position causes the two endpoints to get farther away. This claim generalizes to 3D motions of an initially planar and convex chain that only open the joint angles [O'R00]. Our continuous blooming of the source unfolding suggests the following related problem, phrased in terms of instantaneous motions like Cauchy's Arm Lemma (but which would immediately imply the same about continuous motions):

Open Problem 1 Consider the chain of faces visited by a shortest path on the surface of an arbitrary convex polyhedron. If we open each dihedral angle between consecutive faces, carrying the edges of the shortest path along with their containing faces, does the resulting path always avoid self-intersection?

Indeed, we might wonder whether every dihedral-monotonic blooming of the source unfolding avoids intersection. This problem is equivalent to the following generalization of Open Problem 1.

Open Problem 2 Consider two shortest paths from a common point s to points t and t' on the surface of an arbitrary convex polyhedron, and consider the two chains of faces visited by these shortest paths. If we open each dihedral angle between consecutive faces in each chain, and transform the edges of the two shortest paths along with their containing faces, do the resulting paths always avoid intersecting each other and themselves?

We were unable to resolve either open problem using our techniques, but it remains an intriguing question whether these analogs of Cauchy's Arm Lemma underlie our positive results.[1]

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[1] On the other hand, it is known that the faces of a convex polyhedron intersected by a shortest path can overlap, even when fully developed; the example is essentially [DO07, Fig. 24.20, p. 375] (personal communication with Günter Rote).