Continuous Blooming of Convex Polyhedra*

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Abstract. We construct the first two continuous bloomings of all convex polyhedra. First, the source unfolding can be continuously bloomed. Second, any unfolding of a convex polyhedron can be refined (further cut, by a linear number of cuts) to have a continuous blooming.

1. Introduction.

A standard approach to building 3D surfaces from rigid sheet material, such as sheet metal or cardboard, is to design an *unfolding*: cuts on the 3D surface so that the remainder can unfold (along edge hinges) into a single flat non-self-overlapping piece. The advantage of this approach is that many (relatively cheap) technologies—such as NC machining/milling, laser cutters, waterjet cutters, and sign cutters—enable manufacture of an arbitrary flat shape (with hinges) from a sheet of material. As a result, existence of and algorithms for unfolding have been studied extensively in the mathematical literature.

Often overlooked in this literature, however, is the second manufacturing step: can we actually fold the flat shape along the hinges into the desired 3D surface, without self-intersection throughout the motion? Some unfoldings have no such motion [BLS05, Theorem 4]. In this paper, we develop such motions for large families of unfoldings of convex polyhedra. In particular, we establish for the first time that every convex polyhedron can be manufactured by folding an unfolding.

Unfolding. More precisely, an *unfolding* of a polyhedral surface in 3D consists of a set of cuts (arcs) on the surface whose removal result in a surface whose intrinsic metric is equivalent to the interior of a flat non-self-overlapping polygon, called the *development* or *unfolded shape* of the unfolding. The cuts in an unfolding of a convex polyhedron form a tree, necessarily spanning all vertices of the polyhedron (enabling them to flatten) [BDE+03]. All unfoldings we consider allow cuts anywhere on the polyhedral surface, not just along edges. Four general unfolding algorithms are known for arbitrary convex polyhedra: the source unfolding [SS86, MP08], the star unfolding [AO92], and two variations thereof [IOV08, IOV09]. Positive and negative results for unfolding nonconvex polyhedra can be found in [BDE+03, DFO07, O'R08].

Blooming. Imagine the faces of the cut surface in an unfolding as rigid plates, and the (sub)edges of the polyhedron connecting them as hinges. A *continuous blooming* of an unfolding is a continuous motion of this plate-and-hinge structure from the original shape of the 3D polyhedron (minus the cuts) to the flat shape of the development, while avoiding intersection between the plates throughout the motion. In 1999, Biedl, Lubiw, and Sun [BLS05] gave an example of an unfolding of an orthogonal (nonconvex) polyhedron that cannot be continuously bloomed. In 2003, Miller and Pak [MP08, Conjecture 9.12] reported the conjecture of Connelly that every convex polyhedron has a nonoverlapping unfolding that can be continuously bloomed. He further conjectured that the blooming can monotonically open all dihedral angles of the hinges. More recently, Pak and Pinchasi [PP09] describe a simple blooming algorithm for convex polyhedra which they show works for the finite class of Archimedean solids (excluding prisms and antiprisms, by extending existing bloomings of Platonic solids) but fails on other polyhedra. No other nontrivial positive results have been established.

Our results. We prove Connelly's conjecture by giving the first two general algorithms for continuous blooming of certain unfoldings of arbitrary convex polyhedra, which also monotonically open all hinge dihedral angles. Both of our algorithms have a relatively simple structure: they perform a sequence of linearly many steps of the form "open one dihedral angle uniformly by angle α". Thus our algorithms open angles monotonically, uniformly (at constant speed), and one at a time. The challenge in each case is to prove that the motions cause no intersection.

First we show in Section 2 that every unfolding can be refined (further cut, by a linear number of cuts) into another unfolding with a continuous blooming. Indeed, we show that any serpentine unfolding (whose dual tree is a path) has a continuous blooming, and then standard techniques can refine any unfolding into a serpentine unfolding. The full details and proofs of all unfoldings are omitted. Next we show in Section 3 that one particularly natural unfolding, the source unfolding, has a continuous blooming. It is in the context of the source unfolding that Miller and Pak [MP08] described continuous blooming.

Finally we mention in Section 4 an unsolved more-general form of continuous blooming for the source unfolding that resembles Cauchy's arm lemma.

2. Blooming a Refinement of Any Unfolding.

In this section, we show that any unfolding can be refined into an unfolding with a continuous blooming. An unfolding U is a *refinement* of another unfolding U' if the

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cuts in U form a superset of the cuts in U'. Our approach is to make the unfolding serpentine: an unfolding is *serpentine* if the dual tree of the faces is in fact a path. Here "faces" is a combinatorial notion, not necessarily coinciding with geometry: we allow artificial edges with flat dihedral angles. We do require, however, that every two faces adjacent in the dual path form a nonflat dihedral angle; otherwise, those faces could be merged.

We present three increasingly complex serpentine unrolling algorithms: Path-Unroll (Figure 2), Path-TwoStep, and Path-Waltz. The Path-Waltz avoids all forms of touching except that which was present in the original unfolding, while the two simpler algorithms allow the touching of faces and edges.

3. Blooming the Source Unfolding.

We prove that the following algorithm continuously blooms the source unfolding of any convex polyhedron. Figure 1 shows a simple example of the algorithm in action.

Algorithm 1 (Tree-Unroll) *Let T be the tree of faces formed by the source unfolding, rooted at the face containing the source s of the unfolding. For each face in the order given by a post-order traversal of T, uniformly open the dihedral angle between the face and its parent in T until the two faces become coplanar.*

4. Open Problems.

Cauchy's Arm Lemma states that opening the angles of a planar open chain initially in convex position causes the two endpoints to get farther away. This claim generalizes to 3D motions of an initially planar and convex chain that only open the joint angles [O'R00]. Our continuous blooming of the source unfolding suggests the following related problem, phrased in terms of instantaneous motions like Cauchy's Arm Lemma (but which would immediately imply the same about continuous motions):

Open Problem 1 *Consider the chain of faces visited by a shortest path on the surface of an arbitrary convex polyhedron. If we open each dihedral angle between consecutive faces, and transform the edges of the shortest path along with their containing faces, does the resulting path always avoid self-intersection?*

Indeed, we might wonder whether every dihedral-monotonic blooming of the source unfolding avoids intersection. This problem is equivalent to the following generalization of Open Problem 1.

Open Problem 2 *Consider two shortest paths from a common point s to points t and t' on the surface of an arbitrary convex polyhedron, and consider the two chains of faces visited by these shortest paths. If we open each dihedral angle between consecutive faces in each chain, and transform the edges of the two shortest paths along with their containing faces, do the resulting paths always avoid intersecting each other and themselves?*

We were unable to resolve either open problem using our techniques, but it remains an intriguing question whether these analogs of Cauchy's Arm Lemma underlie our positive results.[1]

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[1] On the other hand, it is known that the faces of a convex polyhedron intersected by a shortest path can overlap, even when fully developed; the example is essentially [DO07, Fig. 24.20, p. 375] (personal communication with Günter Rote).

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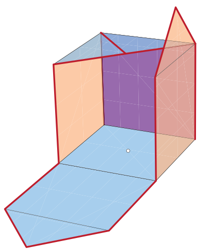

Figure 1. Start of Algorithm Tree-Unroll applied to the source unfolding of a cube, with the source point in the center of the bottom face.

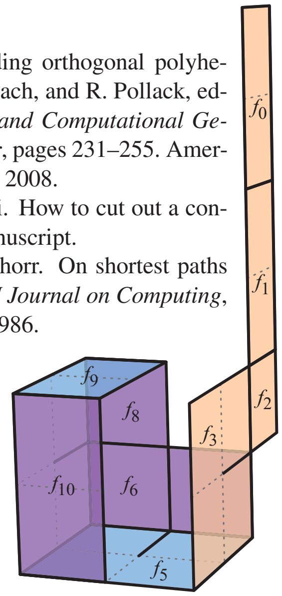

Figure 2. The Path-Unroll algorithm applied to the refined cube unfolding below, after the first $i = 3$ steps.