Polygons Cuttable by a Circular Saw

Erik D. Demaine* Martin L. Demaine* Craig S. Kaplan†

Abstract

We introduce and characterize a new class of polygons that models wood, stone, glass, and ceramic shapes that can be cut with a table saw, lapidary trim saw, or other circular saw. In this model, a *circular saw* is a line segment (in projection) that can move freely in empty space, but can only cut straight into a portion of material. Once a region of material is separated from the rest, it can be picked up and removed to allow the saw to move more freely. A polygon is called *cuttable by a circular saw* if it can be cut out of a convex shape of material by a sufficiently small circular saw. We prove that a polygon has this property precisely if it does not have two adjacent reflex vertices.

1 Introduction

There are many saw tools[1] for cutting a desired shape out of material ranging from soft wood to brittle ceramic, glass, and stone. Cutting out a complex nonconvex shape requires careful maneuvering of the saw, and for many devices is simply impossible because turns cannot be made. For soft materials, the primary example being wood, this problem is easily solved with such devices as the scroll saw and band saw, which permit cutting along a bounded-curvature path because the cut made is slightly wider than the blade. Another example is cutting (indeed melting) styrofoam with a hot wire, a problem studied by Jaromczyk and Kowaluk [2].

For more brittle material, turns are generally impractical but possible using wire saws, either with a diamond-impregnated wire, or a braided wire dripping with silicon-carbide grit. Such saws are slow and can be expensive even for straight cuts. For stained glass and ceramic tiles it is common to score the material, which leads to the well-studied notion of a *glass cut* or *guillotine cut* [1, 3, 4, 5], but only allows for convex shapes. With lapidary saws, e.g., cutting precious and semiprecious stones for jewelry, traditionally the only nonconvex shape that has been of interest is a heart, but more recently there has been a growing interest in cutting more complex nonconvex shapes.

In this paper we discuss *circular saws* such as the popular table saw. Such a device makes precise cuts in essentially any material, using different blades, and is likely the cheapest and fastest motorized saw available. Its only problem is that turns are impossible, essentially because the projection of the blade is a positive-length line segment, and the width of the cut is equal to the width of the blade. As we will show, even without turns it is possible to cut a wide variety of nonconvex shapes.

We stress that a circular saw cannot make turns even if the blade is small. To illustrate this, let us contrast between two popular saws: the table saw and the scroll saw (see Figure 1). The table saw is a type of circular saw, and thus falls under the umbrella of this paper, whereas a scroll saw does not. The difference between the two saws is illustrated by looking at the blades. The key is that the blades have "teeth" to cut a path wider than the blade itself. A circular saw has teeth all around the circle, and hence the cut-out of the blade is at any moment a rectangle, preventing turns because teeth only cut forward. On the other hand, a scroll saw has teeth only at the front edge, and hence the back of the blade is surrounded by empty space, allowing turning.

Model. More formally, we model a circular saw by concentrating on the projection in two dimensions. A circular saw is thus a line segment, of some length d equal to the diameter of the blade.[2] We start with some open set of material M_0, say the interior of a convex polygon. Our goal is to cut away from this material all but the interior of a contained polygon P.

For convenience, we consider the saw moving while the material remains stationary, although in reality the roles are reversed. The saw is permitted to translate and rotate freely in empty space. On the other hand, when the saw is in contact with the material, it may only move by translating in one of the two directions parallel to the saw itself, and thus the three degrees of freedom are reduced to one. In other words, in addition

*Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada, email: {eddemaine, mldemaine}@uwaterloo.ca

†Department of Computer Science and Engineering, University of Washington, Seattle, Washington 98195-2350, USA, email: csk@cs.washington.edu

[1] We do not consider other cutting tools that puncture the material from above rather than cutting through it, such as lasers and water jets.

[2] We assume that the circular saw is oriented perpendicular to the material; although some saws allow it to be tilted, this is not relevant to our problem.

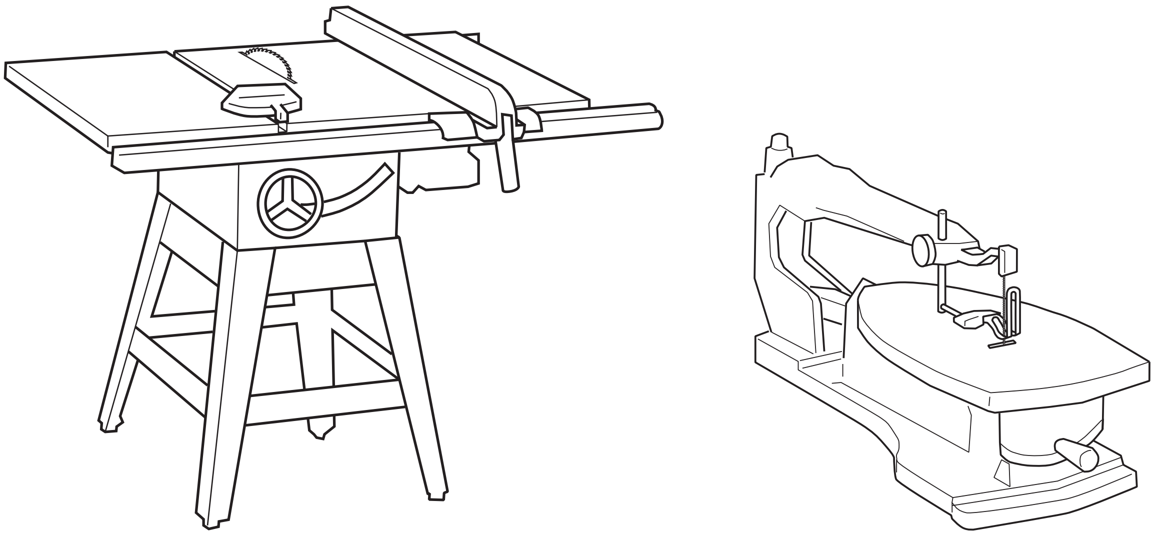


Figure 1: Left: A table saw, which is a type of circular saw, and can only cut straight. Right: A scroll saw, which can cut curves.

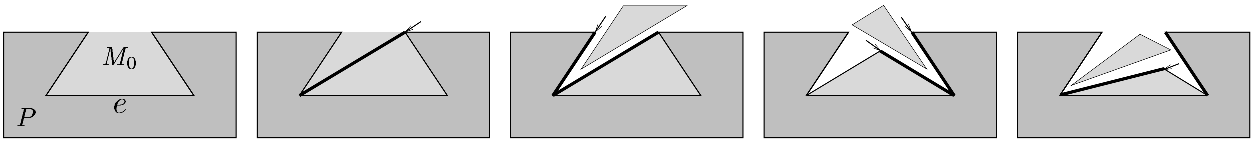


Figure 2: Attempts at cutting all material against an edge e with two reflex endpoints, using a circular saw.

to moving in free space, the saw can either cut all the way through the material, or cut part way in and then retract. Such a motion is called a *cut*; we assume that only a finite number of cuts can be made in a finite amount of time.

The effect of a cut is that it removes material, i.e., subtracts from M any points traversed by the saw. Note that such an operation preserves that the remaining material M is an open set. Thus the *boundary* of M is simply its closure minus itself, i.e, $\overline{M} \setminus M$. In particular, we will refer to the edges of this boundary as *edges* of M.

Finally, once the material M has been disconnected into multiple components, any of the components may be picked up and discarded. (We do not impose that pieces be removed by motions restricted to the plane.) Discarded components become new free space, allowing the saw to move more freely.

A polygon P is *cuttable by a circular saw of diameter* d if it can be cut from its convex hull, or equivalently any convex polygon containing P. More generally, a polygon is *cuttable by a circular saw* if it can be cut with a saw of some (sufficiently small) diameter d. In addition, it is natural to assume that the saw must never slice into the polygon P, even if the cut does not disconnect P.

Note that it is easy to cut any polygon down to a contained convex polygon, because a convex polygon is an intersection of halfspaces, so it can be cut with lines and thus with a circular saw. Thus, for defining cuttability by a circular saw, it is irrelevant which containing convex polygon we start with.

Main Result. In this paper we prove the following theorem characterizing cuttability:

Theorem 1 *A polygon is cuttable by a circular saw precisely if it does not have two adjacent reflex vertices (vertices with interior angle* $> \pi$*).*

Sections 2 and 3 prove necessity and sufficiency of the condition on adjacent reflex vertices for cuttability. In Section 4, we show some examples of cuttable polygons, and methods for making polygons cuttable by a circular saw. We end with several open problems in Section 5.

2 Necessity

Consider an edge e of P whose endpoints are reflex vertices; see Figure 2. In particular, e cannot be an edge of the initial material M_0 because M_0 is convex. We claim that e will never be entirely on the boundary of the remaining material M, no matter how we cut M_0

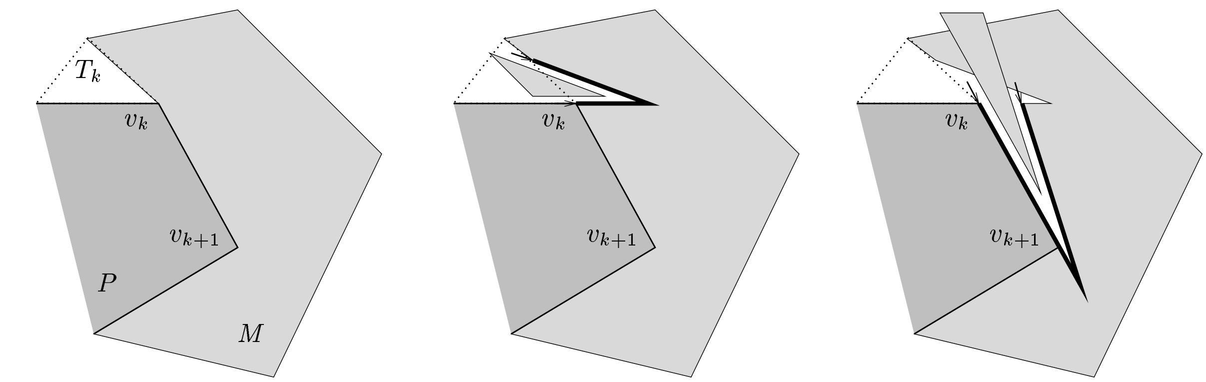


Figure 3: The case of a convex angle v_{k+1}.

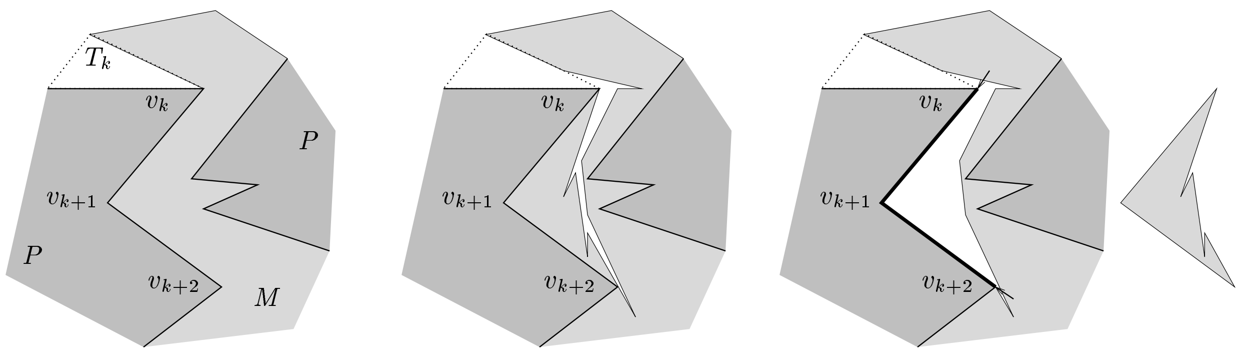


Figure 4: The case of a reflex angle v_{k+1}.

without cutting P. Indeed, we will show the following stronger claim: at all times, and for every point p along e except a finite set of exceptions, there is a neighborhood around p that is fully contained in M.

Initially this claim is true for all points p along e. Apply induction on the number of cuts; in other words, suppose that the claim is true up to some time, and then we make another cut. If this cut starts at a point along e, we claim that it cannot move along the edge starting at that point. This is because the induction hypothesis and the reflexivity of the endpoints of e imply that the saw cannot be oriented against the edge e before coming in contact with the material M. Hence the best the saw could do is cut along a line segment with one endpoint p possibly on the edge e. Every point along e that had a neighborhood in M will still have one, except for p. By induction the result holds.

3 Sufficiency

Now suppose that we have a polygon P that does not have two adjacent reflex vertices. Let v_0 be a vertex on the convex hull of P, and let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of P in cyclic order.

We prove by induction that P is cuttable by a circular saw. The induction hypothesis is that there is a cutting such that the remaining material M, which diminishes from step to step, has the following properties:

1. the chain of polygon edges (v_0, v_1), (v_1, v_2), $\ldots$, (v_{k-1}, v_k) are edges of M;
2. there is a triangle T_k that is disjoint from M, with one vertex at v_k and another vertex along the edge (v_{k-1}, v_k) (if $k \neq 0$), and hence the third vertex is exterior to M and P.

This is clearly true for $k = 0$ because v_0 is on the boundary of the convex hull of P. (If M_0 is not the convex hull of P, we can first cut M down to this convex polygon.) Assume by induction that the hypothesis is true for k. Assume further that v_k is convex; this is true for v_0 and will be preserved by skipping over reflex vertices. We distinguish cases depending on the angle at vertex v_{k+1}.

v_{k+1} **is convex.** This is the easy case; see Figure 3. By maneuvering the saw around T_k, we can cut a slice overlooking the next edge (v_k, v_{k+1}). From this slice we can cut another slice overhanging v_{k+1}. This proves the induction hypothesis for $k + 1$.

v_{k+1} **is reflex.** This case is more difficult; see Figure 4. The idea is to make a slice overhanging v_{k+2}, which is guaranteed to be convex. Unfortunately, v_k and v_{k+2} might not be visible from each other, so it cannot be done in two slices like the previous case. Instead, we make a sequence of possibly several slices before reaching v_{k+2}. At that point, we can cut from v_k to v_{k+1}, and from v_{k+2} to v_{k+1}. This proves the induction hypothesis for $k+2$.

In the end, $k = n$ and the entire boundary of P is on the boundary of M.

4 Examples

Some interesting examples of polygons cuttable and not cuttable by a circular saw are given in Figures 5, 6, 7, and 8. For the last three figures, we wrote a program to automatically modify a given polygon to be cuttable by a circular saw, and applied it to profiles of Trajan, Archimedes, and Hitchcock. Each figure shows the original polygon on the left and the resulting cuttable polygon on the right. Whenever the program finds an edge whose two endpoints are reflex vertices (drawn thick in the figures), it "dents" the edge slightly outward by introducing an additional vertex. Alternatively, we could have introduced a small inwards dent in the middle of each such edge. In either case, we obtain a polygon that is cuttable by a circular saw and arbitrary close to the given polygon (close in terms of the area of the symmetric difference).

Note that for the portraits in profile, few edges need to be modified. This is presumably because the head in profile is made up of a sequence of nearly convex chains: the forehead, top, and back of the head, the curve of the nose, and the chin. A common location for adjacent reflex vertices is in the space between the upper lip and the nose.

5 Conclusion

Several interesting open problems remain:

1. What is the complexity of computing the largest circular saw that can cut a polygon with no adjacent reflex vertices?

2. When the saw parameters are given, it is interesting to consider more general saws in which the cuts can be made with bounded curvature (such as scroll saws).[3] What is the complexity of deciding whether a given polygon (or, indeed, the interior of a simple closed curve) can be cut with a given saw?

3. What is the complexity of computing the optimal sequence of cuts to make a given polygon, where the metric is either the number of cuts or the length of cuts?

4. What can be said about the model in which the cuts must start at infinity, in other words, we cut along (semi-infinite) rays? This problem was posed by Overmars and Welzl in 1985 [5], where they analyzed the case of cutting convex polygons with (doubly infinite) lines so that some metric is minimized, e.g., the total length of the intersections of the cuts with the current polygon. Many additional problems from [5] remain open.

5. Can a similar model be developed for cutting polyhedra in three dimensions? A natural extension is a *handsaw* (or a guillotine that can stop midway) which is modeled by an infinitely long rectangle that can only slice straight into the three-dimensional material.

[3] Note that it is important to fix the saw parameters unlike a circular saw: a scroll saw can cut out any polygon provided the projection of the saw blade is sufficiently short, i.e., the bound on curvature is sufficiently weak relative to the polygon.

References

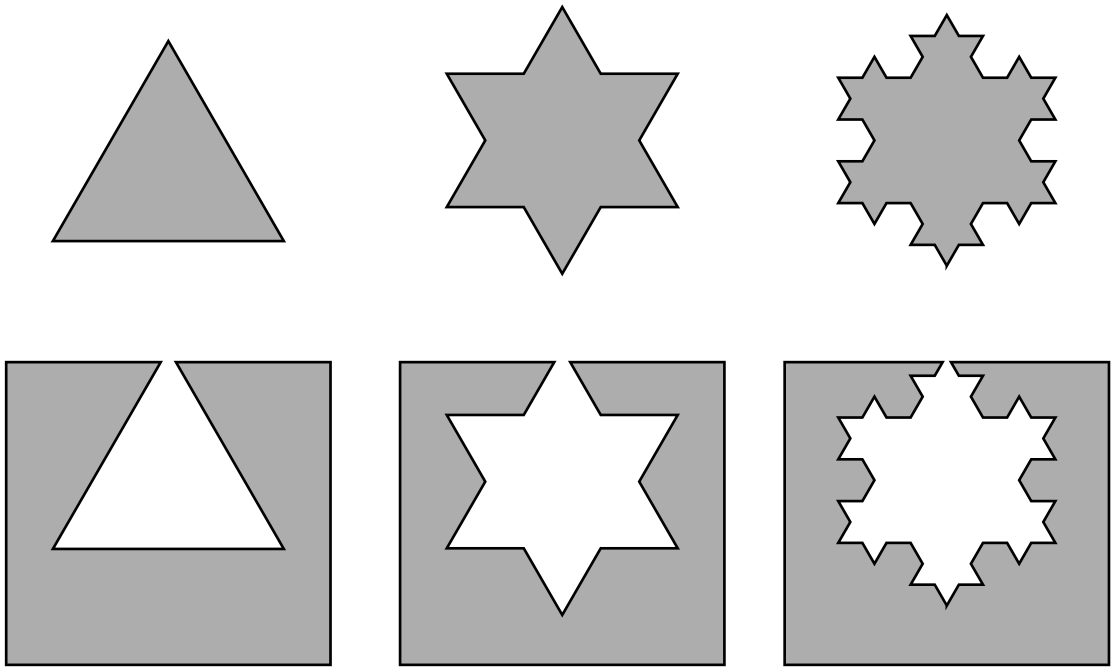
[1] Prosenjit Bose, Jurek Czyzowicz, Evangelos Kranakis, Danny Krizanc, and Dominic Lessard. Near-optimal partitioning of rectangles and prisms. In *Proceedings of the 11th Canadian Conference on Computational Geometry*, Vancouver, Canada, August 1999. `http://www.cs.ubc.ca/conferences/CCCG/elec_proc/fp45.ps.gz`.

[2] Jerzy W. Jaromczyk and Mirosław Kowaluk. The face-wise continuity in hot wire cutting of polyhedral sets. In *Proceedings of the 16th European Workshop on Computational Geometry*, pages 93–97, Eilat, Israel, March 2000.

[3] H. Martini and V. Soltan. Minimum convex partition of polygonal domains by guillotine cuts. *Discrete and Computational Geometry*, 19(2):291–304, 1998.

[4] Joseph S. B. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: A simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems. *SIAM Journal on Computing*, 28(4):1298–1309, 1999.

[5] Mark H. Overmars and Emo Welzl. The complexity of cutting paper. In *Proceedings of the 1st Annual ACM Symposium on Computational Geometry*, pages 316–321, Baltimore, Maryland, June 1985.



(a) (b) (c)

Figure 5: Saw cutting and the Koch snowflake: we attempt to cut the first three iterations of the fractal and their complements. In (a), the triangle can be cut but its negative cannot. In (b), both the positive and negative shapes are cuttable. In (c), the negative can be cut but the positive cannot. It is easy to see that all further iterations behave like (c).

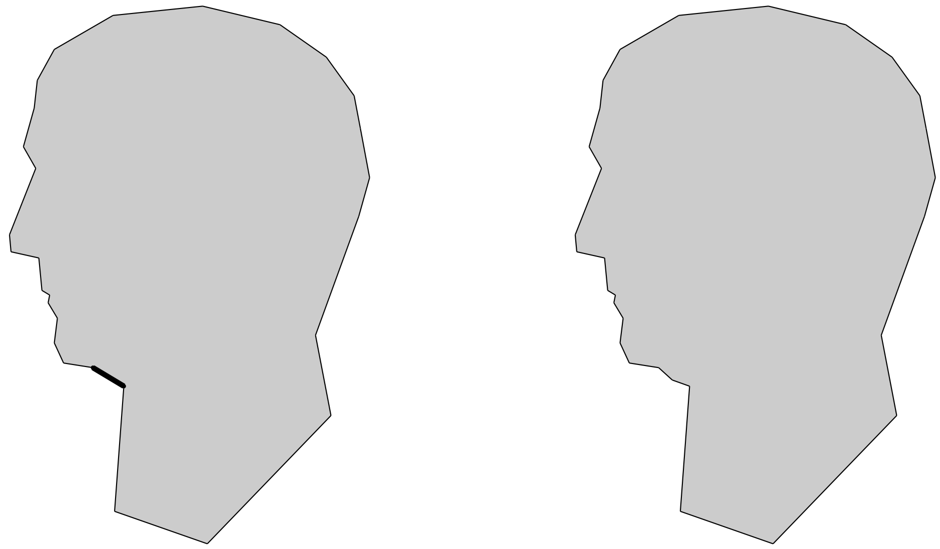


Figure 6: Profile of Trajan (Marcus Ulpius Traianus), Roman Emperor 98–117 AD.

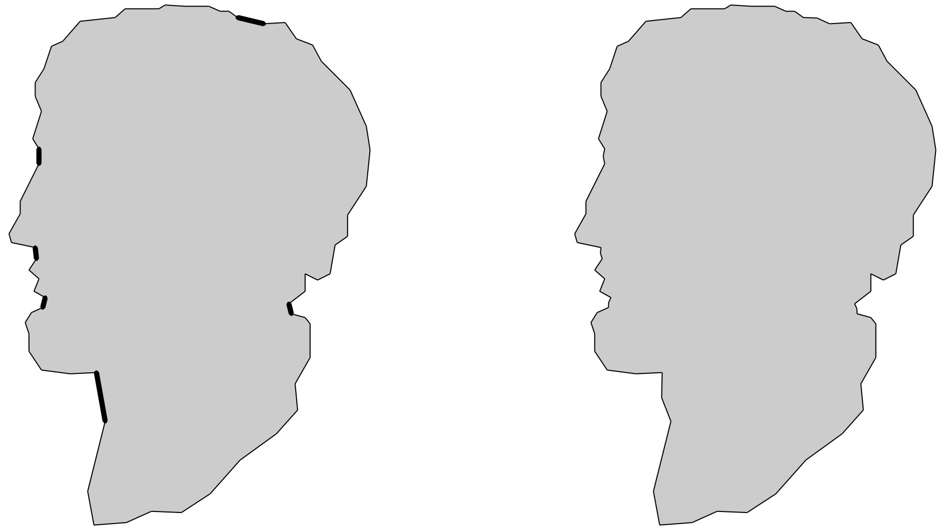


Figure 7: Profile of Archimedes (287–212 BC).

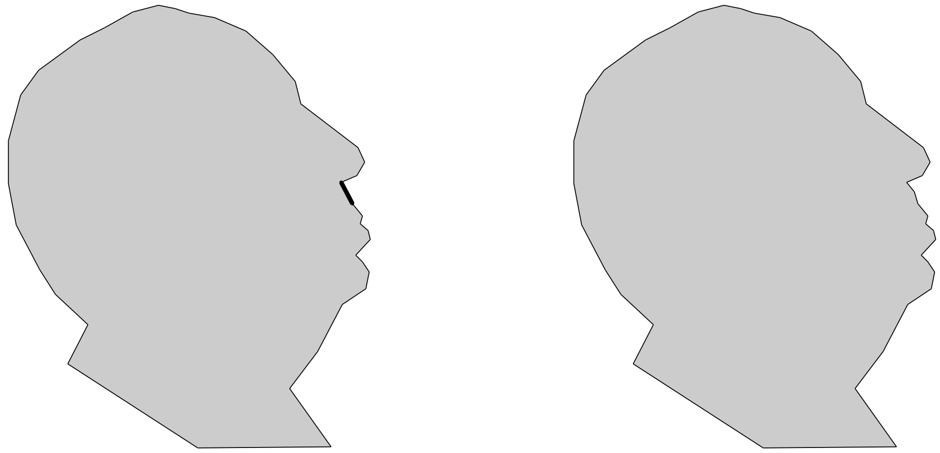


Figure 8: Profile of Alfred Hitchcock (1899–1980 AD).