Reconfiguring Convex Polygons

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Abstract. We prove that there is a motion from any convex polygon to any convex polygon with the same counterclockwise sequence of edge lengths, that preserves the lengths of the edges, and keeps the polygon convex at all times. Furthermore, the motion is "direct" (avoiding any intermediate canonical configuration like a subdivided triangle) in the sense that each angle changes monotonically throughout the motion. In contrast, we show that it is impossible to achieve such a result with each vertex-to-vertex distance changing monotonically.

1 Introduction

This paper is concerned with *linkages* modeled by polygons (primarily in the plane), whose vertices represent hinges and whose edges represent rigid bars. A fundamental question about such linkages is whether it is possible to reach every polygon with the same sequence of edge lengths by motions that preserve the edge lengths. Several papers have shown that the answer to this question is yes for various types of polygons; we call this a *universality* result. If edges are allowed to cross each other, then this is true in every dimension [11, 12]. If edges are not allowed to cross, universality does not hold in general for polygons in 3-D [2, 5], but has been shown for polygons in the plane and motions in 3-D [1, 2], for polygons and motions in the plane [8], for polygons in 3-D with simple projections [4], and for all polygons in 4-D and higher dimensions [7].

All of these papers show universality by proving that every polygon can be *convexified*, that is, moved to a convex (planar) polygon while preserving edge lengths. Convex polygons are used as an intermediate state; because motions can be reversed and concatenated, all that remains is to show that a convex polygon can be moved to every other convex polygon with the same counterclockwise sequence of edge lengths. This fact is established in [11] when edges are allowed to cross. No proof has been published for the case in which edges cannot cross.

The basic idea in the proof in [11] of universality of convex polygons is to show how to reconfigure every convex polygon into another intermediate state, a "canonical triangle." In this paper, we show that this intermediate state can be avoided. Specifically, a convex polygon can be moved into any other convex polygon with the same counterclockwise sequence of edge lengths in such a way that each vertex angle varies monotonically with time (either never increasing or never decreasing). In this sense, the motion goes directly from the source to the destination. Our motion is also of the simplest type possible [3]: it can be decomposed into a linear number of *moves*, each of which changes only four joint angles at once.

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation that we will use throughout the paper. Section 3 proves the main theorem mentioned above, using an old lemma of Cauchy and Steinitz. We conclude in Section 4 by showing an example in which a different type of monotonicity cannot be achieved.

2 Notation

For a polygon P, we denote its vertices by $v_1, \ldots, v_n$ in counterclockwise order, its edges by $e_i = (v_i, v_{i+1})$, and its edge lengths by $\ell_i = |e_i| = |v_i - v_{i+1}|$. Throughout, index arithmetic is modulo n.

A *convex configuration* of edge lengths (positive real numbers) $\ell_1, \ldots, \ell_n$ is a convex polygon with those edge

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lengths in counterclockwise order. A well-known result characterizes the edge lengths for which convex configurations exist:

Lemma 1 (Lemma 3.1 of [11]) *The edge lengths $\ell_1, \ldots, \ell_n$ admit a convex configuration precisely if $\ell_i \leq \sum_{j \neq i} \ell_j$ for all i.*

A *motion* or *reconfiguration* is a continuous function from the unit interval $[0, 1]$ (representing time) to a configuration, where each *configuration* is a polygon with the same counterclockwise sequence of edge lengths. An *angle-monotone motion* is a motion in which each vertex angle is a monotone function in time.

In the following, we split our results into two components: theorems give the theoretical result, and propositions give the additional computational result.

3 Reconfiguring between Two Convex Configurations

Consider two convex configurations C and C' of the same sequence of edge lengths. We think of C as the source configuration and C' as the destination configuration. Label each angle of C by $+$ if it needs to get bigger in order to match the corresponding angle in C', by $-$ if it needs to get smaller, or by 0 if they already match.

This set up is exactly what arises in the proof of Cauchy's theorem about the rigidity of convex polyhedra [6, 9], except that in Cauchy's application the polygon is on the sphere. His key lemma about alternations in such $+, -, 0$ labelings is what we need as well. Cauchy's original proof of this lemma (in 1813) had an error, noticed and corrected over a century later by Steinitz in 1934 [15].

Lemma 2 (Cauchy-Steinitz Lemma) *In a $+, -, 0$ labeling that comes from two distinct convex configurations, there are at least four sign alternations.*

Proof (Sketch): Because the configurations are distinct, not all labels are 0. By circularity, the number of alternations between $+$ and $-$ (ignoring 0's) is even. It cannot be zero, because there is no motion of any polygon that increases or decreases all angles. It cannot be two, because then there is a chain of increasing angles and a chain of decreasing angles; the former chain specifies that the ends of the chain should get further apart, whereas the latter chain specifies the opposite. It is this last part of the argument that needs careful analysis; for details, see [15] for Steinitz's original (complicated) proof, [9] for a simpler proof due to Issac J. Schoenberg, or [13] for another elementary proof. □

The idea is to take vertices v_i, v_j, v_k, v_l in cyclic order around the polygon, whose angles are labeled $+, -, +, -$ in that order, and flex the quadrangle defined by those vertices until one angle matches the desired value in C'. See Figure 1.

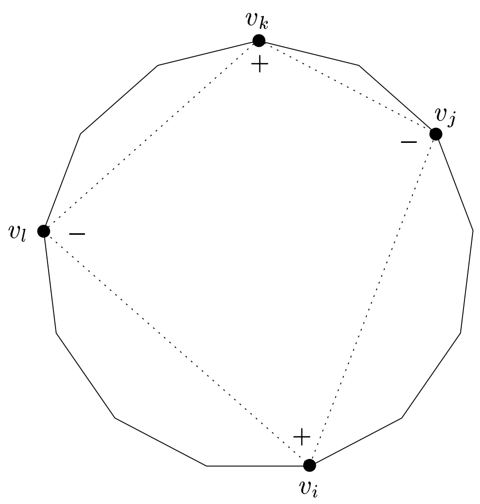

Figure 1: Applying a quadrangle motion to a convex polygon by taking vertices labeled $+, -, +, -$ in that order.

Now we need a lemma about reconfiguring convex quadrangles:

Lemma 3 *Given a convex quadrangle v_1, v_2, v_3, v_4, there is a motion that decreases the angles at v_1 and v_3, and increases the angles at v_2 and v_4. The motion can continue until one of the angles reaches 0 or π.*

Proof: We consider the following viewpoint: v_1 is pinned to the plane, and v_3 moves along the directed line from v_1 to v_3 (see Figure 2). The motions of v_2 and v_4 are determined by maintaining their distances to v_1 and v_3. Applying Euclid's Proposition I.25 [10] to triangle v_1, v_2, v_3, because $|v_1 - v_3|$ is increasing, so is the angle at v_2. Similarly, the angle at v_4 is increasing throughout the motion. Because no angle goes past 0 or π, we maintain a convex quadrangle throughout the motion, so by the Cauchy-Steinitz lemma (Lemma 2), there must be at least four sign alternations when compared to any future quadrangle we will visit. This proves that the angles at v_1 and v_3 are decreasing throughout the motion. □

We can prove our main theorem:

Theorem 1 *Given two convex configurations C, C' of the same edge lengths $\ell_1, \ldots, \ell_n$, there is an angle-monotone motion from C to C' that involves $O(n)$ moves each of which changes only four vertex angles at once.*

Proof: Consider configuration C. By Lemma 2, we can find vertices v_i, v_j, v_k, v_l in cyclic order around the

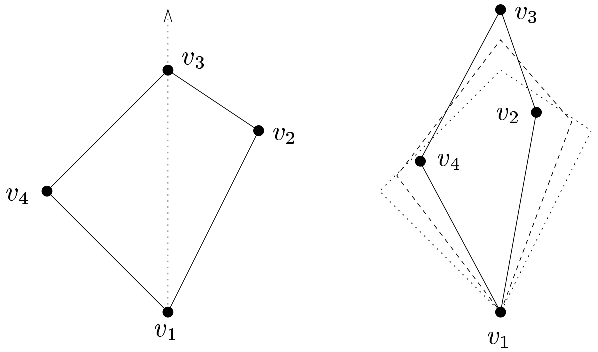

Figure 2: Moving a convex quadrangle as in Lemma 3.

polygon, whose angles are labeled $+, -, +, -$ in that order; see Figure 1. By specifying the subchains between these vertices to move rigidly, we obtain a convex quadrangle. Move this quadrangle according to Lemma 3 until one of the four angles matches the angle in C'. (No angle will ever reach 0 or π because of our stopping condition.) Repeat this process until all angles match. The result is a sequence of motions from C to C', with at most n moves, because each motion changes the label of an angle from $+$ or $-$ to 0, and that label persists. $\square$

Proposition 1 *Computing the motion in Theorem 1 can be done in $O(n)$ time on a pointer machine with real numbers.*

Proof: The first part is to maintain the vertices of the quadrangle, v_i, v_j, v_k, v_l, throughout the motion. We maintain four consecutive blocks I, J, K, L of the same sign; specifically, we maintain the first and last vertex in each block. This can be found initially in linear time by scanning along the polygon's vertices in order. The desired vertices v_i, v_j, v_k, v_l are identified with the first vertex in the corresponding block. When the label of one of them switches to 0, it and the block's first vertex advance to the next element in the block. If this was the last element (the block is empty), we make the following modifications. If I becomes empty, we advance it to the block of $+$'s after L. Similarly, if L becomes empty, it retreats to the block before I. If K becomes empty, it advances to the block after L, the blocks J and L merge to produce a new J, and L advances to the block after K. The case of J becoming empty is symmetric.

The second part is to apply the quadrangle motions from Lemma 3. This involves computing the time at which the quadrangle motion stops, and then updating the coordinates. These computations can be done analogous to Lemma 7 of [3]. Basically, we compute the times at which each angle would match the desired angle in C', and take the minimum of these times. At worst, each time can be computed by solving a degree-four polynomial, which reduces to an arithmetic expression involving square and cube roots. $\square$

4 Conclusion

We have shown that an angle-monotone motion between any two convex configurations of a common sequence of edge lengths can be computed in linear time. An interesting consequence is that any polygon can be moved to a unique *inscribed* configuration [14], in which the vertices are cocircular, a natural generalization of regular polygons.

It is interesting to note that we cannot hope for a *distance-monotone* motion between any two convex polygons, in which every distance between a pair of vertices varies monotonically with time (like [8]). An example is shown in Figure 3. Because the dotted lines are the same length in both configurations, these distance must be preserved throughout the motion; in other words, the chains v_1, v_2, v_3 and v_4, v_5, v_6 must move rigidly. The problem is thus reduced to moving a quadrangle v_1, v_3, v_4, v_6, which can be moved in only two different ways. Only one motion decreases $|v_1 - v_4|$ and increases $|v_3 - v_6|$ as desired, but then the distance $|v_2 - v_5|$ increases and later decreases. Specifically, the distance in the middle configuration is more than 0.6% larger than the (equal) distances in the left and right configurations.

Finally, we report on a related result about reconfiguring convex polygons. This paper has shown how to reach any convex configuration from any other by using four-joint rotations in the plane. In the full version of this paper, we will also show that a motion is possible using a sequence of *restricted* moves through *three dimensions*. Namely, a *pivot* divides the chain into two subchains, keeping one rigid, and rotating the other by some amount around the line through the two endpoints of the subchains. As a consequence of proving that convex polygons can be reconfigured arbitrarily via pivots, we obtain universality results about reconfiguring general (nonconvex) polygons via pivots.

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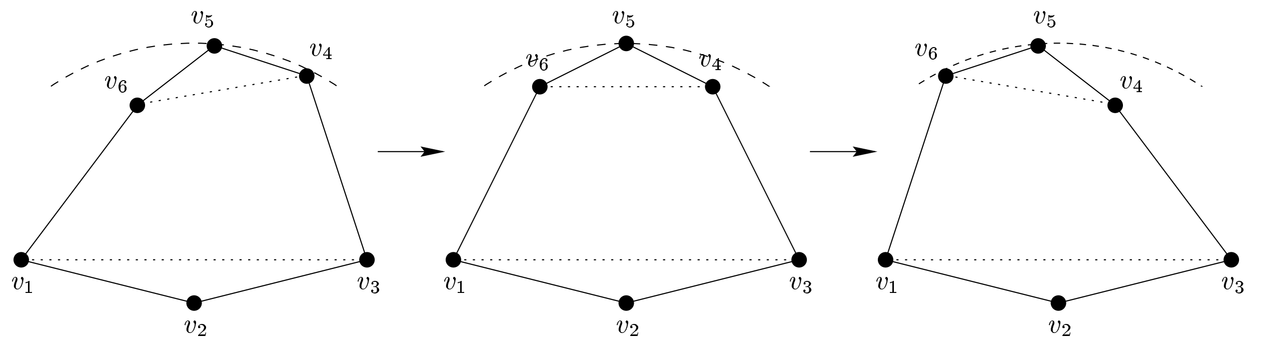

Figure 3: (Left and right) An example for which a distance-monotone motion is impossible. (Middle) The transition between $|v_2 - v_5|$ increasing and decreasing.

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