Open Problems from CCCG 2008

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The following is a list of the problems presented on August 14, 2008 at the open-problem session of the *20th Canadian Conference on Computational Geometry* held in Montréal, Québec, Canada.

Dark Points in Mirror Polygons
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Tokarsky [Tok95] and Castro [Cas97] constructed a polygon, all of whose edges are mirrors, with the property that a light placed at a particular interior point does not illuminate the entire interior, but rather leaves one dark point. The following conjecture remains:

Conjecture [OP01]: In any mirror polygon with one interior point light source, the set of dark points has measure zero.

References

[Cas97] D. Castro. Corrections. *Quantum* 7:42, January 1997. See http://mathworld.wolfram.com/IlluminationProblem.html.

[OP01] Joseph O'Rourke and Octavia Petrovici. Narrowing light rays with mirrors. In *Proc. 13th Canad. Conf. Comput. Geom.*, pages 137–140, 2001.

[Tok95] George W. Tokarsky. Polygonal rooms not illuminable from every point. *Amer. Math. Monthly*, 102:867–879, 1995.

Segment Mirrors
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A related conjecture remains outstanding:

Conjecture [OP01]: No finite collection of disjoint, double-sided segment mirrors can trap the light from any one point source.

It seems most natural to treat the mirrors as open segments, to avoid endpoint effects, but the segments should be disjoint when closed.

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Swimming in an Equilateral Triangle Lake
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A swimmer falls overboard in an equilateral triangle lake in a fog. If the edge length of the triangle is 1, what is the shortest length swimming path that ensures the swimmer will reach the shore? Swimming one unit straight in any direction certainly reaches the shore. The problem was originally posed in 1963. Besicovitch showed shortly afterward that a 3-link chain of length $3\sqrt{21}/14 = 0.98198\ldots$ suffices [Bes65].

Update. The problem was recently solved [CM06]: Besicovitch's chain is indeed optimal.

References

[Bes65] A. S. Besicovitch. On arcs that cannot be covered by an open equilateral triangle of side 1. *Math. Gazette*, 49:286–288, 1965.

[CM06] P. Coulton and Y. Movshovich. Besicovitch triangles cover unit arcs. *Geometriae Dedicata*, 123(1):79–88, 2006.

Simplices Containing the Origin
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Call two simplices S_1 and S_2 in $\mathbb{R}^d$ *connected* if they both contain the origin and S_1 and S_2 share all vertices except one. (In other words, S_2 can be made from S_1 by replacing one vertex with another point, while still containing the origin.) Is it possible, for any two simplices S and S' both containing the origin, to find a sequence $S = S_1, S_2, \ldots, S_k = S'$ such that each S_i is connected to S_{i+1}? This is known to be true in 2D, but $d \geq 3$ is open.

Update. The following short proof was discovered by Donald Sheehy. Let V be the set of vertices of S_b and S_t. A Gale transformation maps V to a point set P in $\mathbb{R}^{d+1}$ where a simplex in V containing the origin corresponds to a $(d+2)$-dimensional face of the convex hull H of P, and two connected

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simplices correspond to two adjacent faces of H. The correctness of the claim follows by observing that the graph built on the adjacent faces of H is connected. See [HRZ04] and [K04] for more details on the tools used above.

References

[HRZ04] Martin Henk, J. Richter-Gebert, and G. M. Ziegler. Basic properties of convex polytopes. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 355–381. CRC Press, Inc., 2004.

[K04] Gil Kalai. Polytope skeletons and paths. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 455–476. CRC Press, Inc., 2004.

Median Interpoint Distance
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Consider the $\binom{n}{2}$ interpoint distances determined (in general) by n given points in $\mathbb{R}^d$. Can the median interpoint distance be found in $O(n \log n)$? Of course it can be found in $O(n^2)$ for fixed d. It is known how to find the kth smallest interpoint distance in subquadratic time, specifically $O(n^{4/3} \log^{8/3} n)$ [AASS93] randomized time. The same question may be asked about the mean interpoint distance.

Update. Pat Morin noted that Timothy Chan's result [Cha01] achieves randomized time $O(n \log n + n^{2/3} k^{1/3} \log^{5/3} n)$ for the median interpoint distance.

References

[AASS93] Pankaj K. Agarwal, Boris Aronov, Micha Sharir, and Subhash Suri. Selecting distances in the plane. *Algorithmica*, 9:495–514, 1993.

[Cha01] T. M. Chan. On enumerating and selecting distances. *Internat. J. Comput. Geom. Appl.*, 11:291–304, 2001.

Even-Ranked Sum
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Given n points on a line specified by real numbers, the sum of the $n/2$ largest numbers can be found in linear time. Find the sum of the even-ranked numbers: the 2nd-smallest, 4th-smallest, 6th-smallest, etc. Is $\Omega(n \log n)$ a lower bound?

Update. An $\Omega(n \log n)$ lower bound was established in [MST09].

References

[MST09] Marc Mörig, Michiel Smid, and Jan Tusch. An $\omega(n \log n)$ lower bound for computing the sum of even-ranked elements. Unpublished manuscript, January 2009.

Matrices
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Given $n \times n$ matrices A and B, can you decide whether $A^2 = B$ and/or $A^{-1} = B$ faster than the time needed to multiply two matrices?

Update. Pat Morin pointed out that Freivalds' technique [Fre77] can solve this problem by a randomized algorithm, as follows. Let r be a random vector in $\{1, 2, \ldots, k\}^n$. Compute $F_1 = (rA)A$ and $F_2 = rB$, each in $O(n^2)$ time. If $A^2 \neq B$, then the probability that $F_1 = F_2$ is at most $1/k$.

References

[Fre77] Rusins Freivalds. Probabilistic machines can use less running time. In B. Gilchrist, editor, *Proceedings of IFIP Congress 77*, pages 839–842. North-Holland, 1977.

Best-case 3D Hull
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Does there exist an ordering of n points in 3D so that their convex hull can be computed in linear time? This question was originally asked by Jack Snoeyink in 1997. There was considerable discussion at the conference of the right form this question should take in order to avoid trivial solutions.

Update. Following a pointer suggested by Pat Morin, the poser solved the problem in [Bar08] by extending work of Snoeyink and van Kreveld [SvK97]. Another solution was obtained by the poser with Timothy Chan and Peyman Afshani, obtaining "order-oblivious instance-optimal" algorithms for the convex hull and a few other problems.

References

[Bar08] Jérémy Barbay. Adaptive (analysis of) algorithms for convex hulls and related problems. Technical Report Technical Report TR_DCC-2008-017, Universidad de Chile, December 2008.

[SvK97] Jack Snoeyink and M. van Kreveld. Good orders for incremental (re)constructions. In *Proc. 13th Annu. ACM Sympos. Comput. Geom.*, pages 400–402, 1997.

Pleat Folding
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How many simple folds does it take to fold a given 1D mountain-valley pattern? More specifically, we are given a set of creases parallel to the short side of a long rectangular strip of paper, with each crease specified either mountain (M) or valley (V), and the goal is to fold the strip to obtain exactly the given pattern, minimizing the number of folds. Each fold is along one of the creases, and unfolds are free. Each crease may be folded several times, as long as the final fold creases it in the correct direction.

An easy lower bound is $\lg n$ (folding repeatedly in half), and an easy upper bound is n (fold a crease, unfold, repeat). The poser can achieve $n/2 + \log n$ by folding the majority, and with some effort, $n/3+$ polylog n. The alternating $(MV)^n$ pattern is an interesting special case.

Update. Ito et al. [IKI+09] describe the $n/2+\log n$ upper bound and prove an $\Omega(n/\log n)$ lower bound for general patterns, and for the $(MV)^n$ pattern, prove an $O(n^\varepsilon)$ upper bound for any $\varepsilon > 0$. Cardinal et al. [CDD+09] prove a matching upper bound of $O(n/\log n)$ for general patterns, and for the $(MV)^n$ pattern, prove a tighter upper bound of $O(\text{polylog}\, n)$.

References

[CDD+09] Jean Cardinal, Erik Demaine, Martin Demaine, Shinji Imahori, Stefan Langerman, and Ryuhei Uehara. Algorithmic folding complexity. Manuscript, 2009.

[IKI+09] Tsuyoshi Ito, Masashi Kiyomi, Shinji Imahori, and Ryuhei Uehara. Complexity of pleat folding. In *Abstracts from the 25th European Workshop Comput. Geom.*, March 2009.

Arrangement Cells
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Suppose we are given a set of n hyperplanes in $\mathbb{R}^d$. The goal is, for each of the $\Theta(n^d)$ cells of the arrangement A induced by the hyperplanes, to list all the hyperplanes on its boundary in any order, using less than $\Theta(n^d)$ space (ideally $O(dn)$) in addition to the input (not counting the write-only output). This constraint especially rules out constructing the arrangement A. It would be fine to repeat cells; the overall runtime, however, should be polynomial in n.

Regular Triangulation
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A regular triangulation is an orthogonal projection of a lower hull of points. An example of a non-regular triangulation is Schönhardt's polyhedron. Edelsbrunner found sufficient conditions (acyclicity criteria) for nonregularity [Ede90]. The decision problem—is this triangulation regular?—can be solved by linear programming. Can it be determined more quickly whether a given triangulation is regular? An equivalent formulation is to determine whether a given triangulation is the orthogonal projection of a convex polytope.

The difficulty here is not so much algorithmic. Rather it is that all the nonregular triangulations seem to be variations on the Schönhardt polyhedron. Does the twisting that happens with Schönhardt and similar triangulations completely characterize non-regularity? If so, what characterizes this twisting?

References

[Ede90] Herbert Edelsbrunner. An acyclicity theorem for cell complexes in d dimensions. *Combinatorica*, 10(3):251–260, 1990.

Visibility Polygon Morphing
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Given a simple polygon P and its internal vertex visibility graph G (an edge connecting two vertices if the segment is nowhere exterior to P), can the vertices of P be moved continuously (one at a time or simultaneously) along paths so that (a) simplicity of the polygon is maintained at all times, and

(b) visibility only increases. In other words, if P' is a later version of P, then its visibility graph G' contains a superset of the edges in G. This process ends with a convex polygon whose visibility graph is the complete graph.

Update. Oswin Aichholzer, Gelasio Salazar, and their students proved (at the Mexican Workshop on Computational Geometry) that the morphing is possible for monotone polygons. They also found an example that shows that the problem cannot be solved by moving one vertex at a time while strictly increasing the visibility graph in each move. The context of the posed problem is explained in [DSSW09], which also extends the monotone result to star-shaped polygons.

References

[DSSW09] Satyan L. Devadoss, Rahul Shah, Xuancheng Shao, and Ezra Winston. Visibility graphs and deformations of associahedra. arXiv:0903.2848, March 2009.

Packing an Equilateral Triangle with Equal Disks

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What is the densest packing of equal disks in an equilateral triangle? Let $T(N)$ be the minimum side length for N points in an equilateral triangle with pairwise distances at least 1. It is clear that $T\binom{n+2}{2} = n$ by "the appropriate triangular subset of the hexagonal packing of the disks (well known to pool players in the case of n=15)" [GL95].

1. What is the answer for one fewer point? The conjecture is that the answer is still n: $T\left(\binom{n+2}{2} - 1\right) = n$. Ron offered a \$500 reward for settling this question either way.
2. What is the answer for one more point? The conjecture is that side length more than n is needed, say, $T\left(\binom{n+2}{2} - 1\right) = n + \frac{1}{5}$. Again Ron offered \$500 for a resolution.

See [GL95].

References

[GL95] Ronald L. Graham and Boris D. Lubachevsky. Dense packings of equal disks in an equilateral triangle: from 22 to 34 and beyond. *Electr. J. Combinatorics*, 2, 1995.