3rd CGC Workshop on Computational Geometry

FOLDING ANY SILHOUETTE FROM A STRIP

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Joint work with

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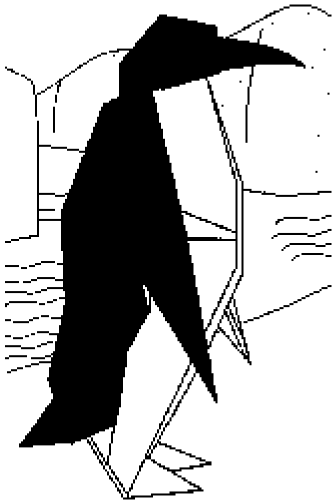
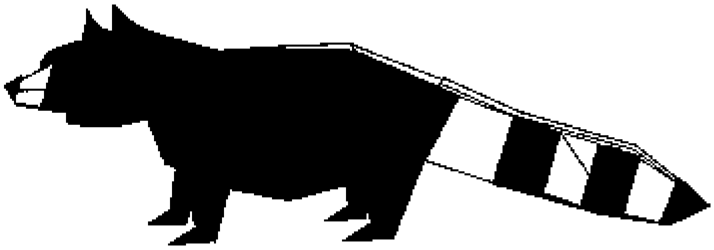
http://daisy.uwaterloo.ca/~eddemain/talks/CGC98/

The Problems

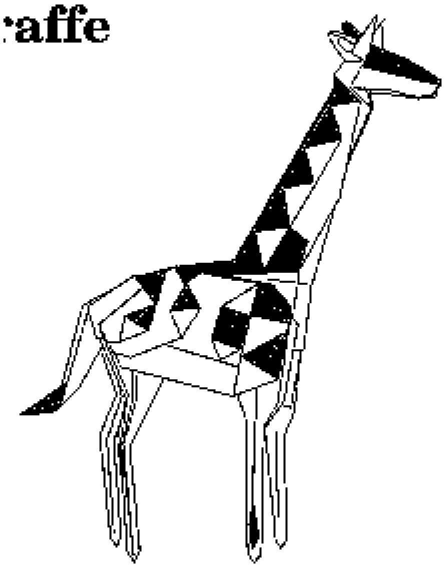
- **Silhouette question** (Bern & Hayes, SODA'96):
  Is every polygon the silhouette of a flat origami?
  - More generally, what about polygonal regions (polygons with holes)?
- **2-color origami problem**:
  Construct a given 2-color pattern with bicolor paper
  - 2-color pattern: Polygonal region partitioned into subregions, each assigned one of two colors
- **Gift wrapping question**:
  Can every (nonconvex) polyhedron be wrapped by a sufficiently large piece of paper?
  - Perhaps assigned a 2-color pattern on each face
- **Theorem**: All of this can always be done
  - Indeed, the amount of paper required can approximate the surface area of the polyhedron

2-Color Flat Origamis

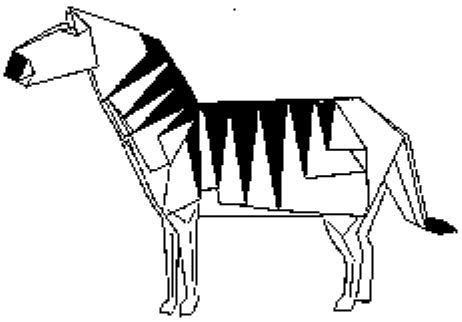




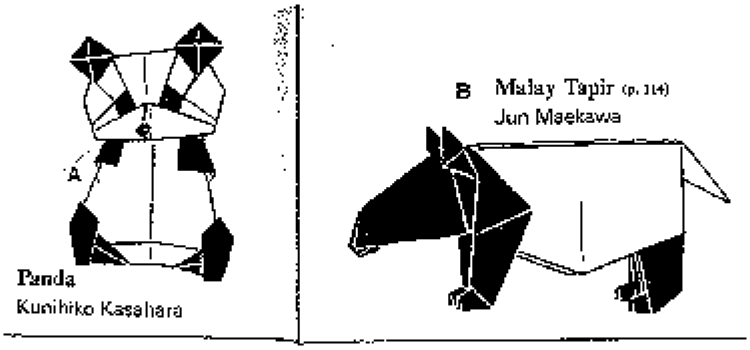
:affe



Zebra



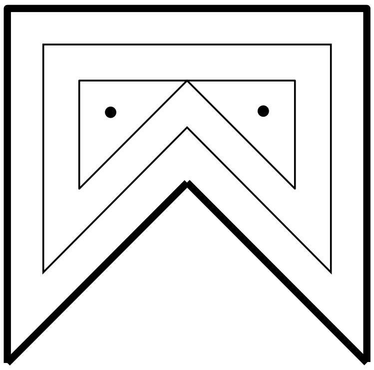
The zebra is one of the most well-recognized and highly



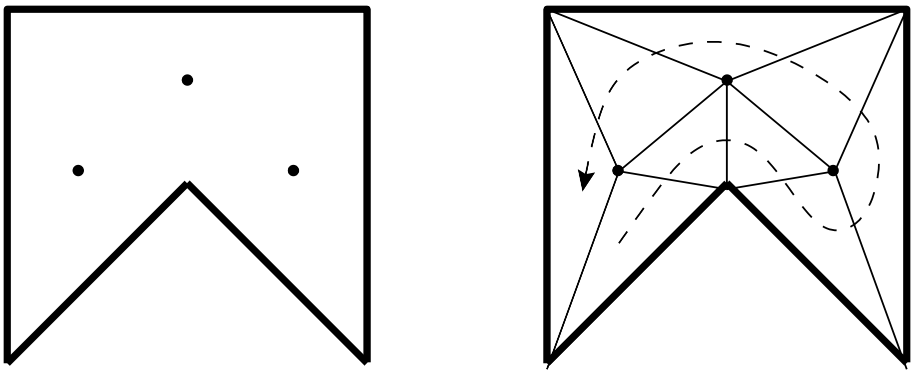
Overview of Approaches

Basic idea: Use a "strip" of paper

1. **Ring approach**
   - Based on ring decomposition from straight skeleton



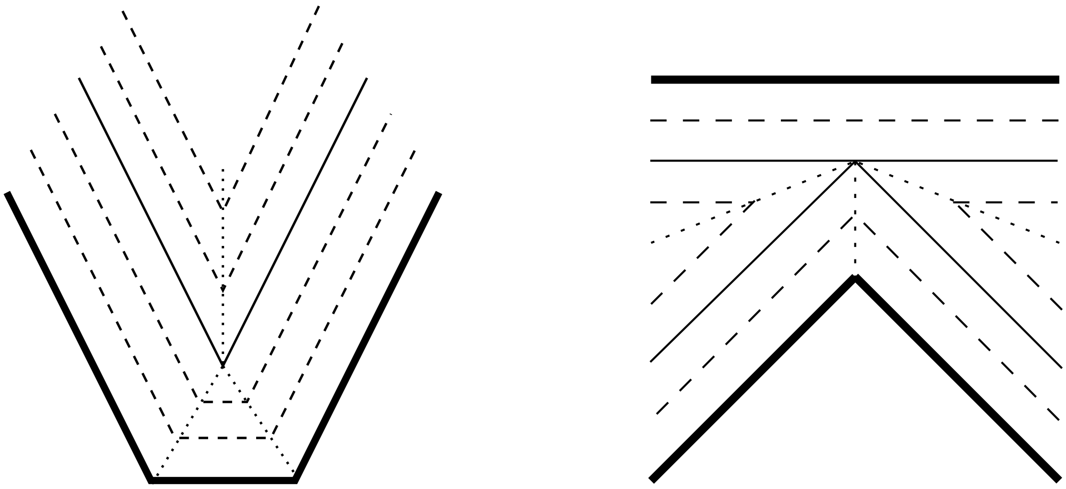
2. **Triangle approach**
   - Add a few points to interior of polygonal region
   - Triangulate so that there is a path that visits each triangle precisely once
   - Cover each triangle in turn along the path



Straight Skeleton

(Aichholzer & Aurenhammer 1996)

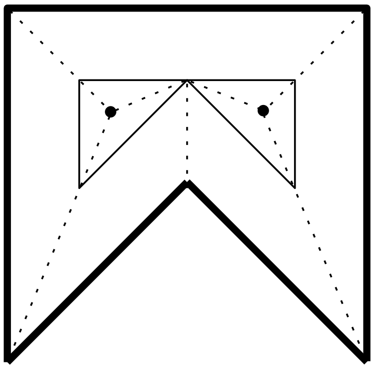
- **Straight skeleton** is the trajectories of the boundary vertices during a shrinking process



  - Inset all boundary edges simultaneously
  - At any point in time:
    - Inset edge is parallel to original edge
    - Distance between them along a perpendicular is the same over all edges
  - When boundary becomes nonsimple, recursively shrink each subregion

Ring Decomposition

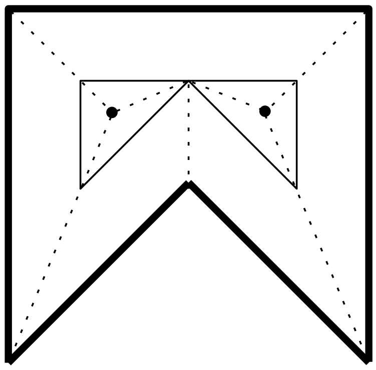
- Regions outlined by the boundary at times of topological change are called **skeleton rings**



- Properties of rings:
  - Two "walls" that are parallel except at turns
  - Constant width all around
- Skeleton rings can be computed in $O(n^2)$ time from the straight skeleton
- Straight skeleton can be computed in $O(n^{11/7})$ time (Eppstein and Erickson 1998)

Basic Algorithm

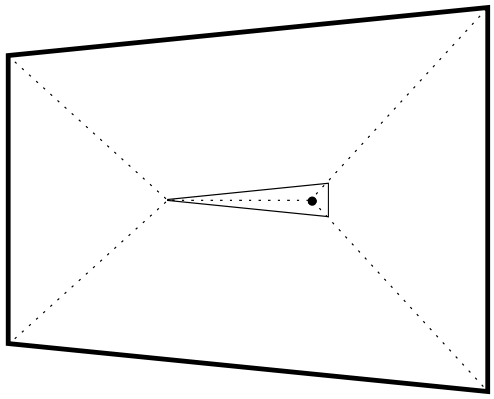
- Two components:
  1. How to construct a skeleton ring
  2. How to bridge between two skeleton rings with a possible color change
- Depth-first traversal of skeleton rings



  – Whenever you visit a skeleton ring for the first time, construct it in entirety
  – Then construct part of the skeleton ring to "walk" adjacent to the next skeleton ring
  – Then bridge between them

Strip Rings

- Subdivide skeleton rings into **strip rings**, i.e., a ring with the same width as the strip of paper
- By making the strip thinner than all skeleton rings, can cover each skeleton ring using strip rings
- This may lead to very thin strips:

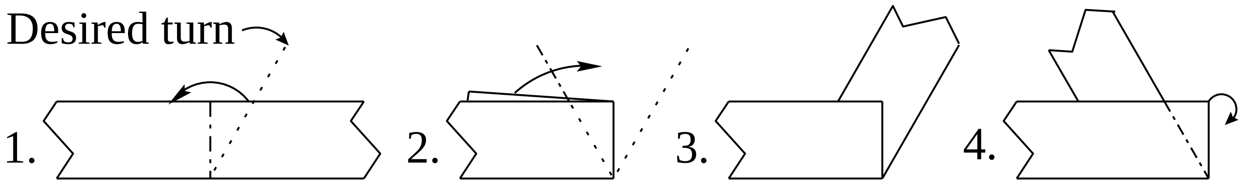


- Alternatively, can generalize the notion of rings to allow nonsimple walls
  - Then the minimum allowed strip width is at least

$$\text{minimum feature size} \cdot \sin\left(\frac{\text{sharpest angle}}{2}\right)$$

How to Build a Strip Ring

- Turn gadgets:

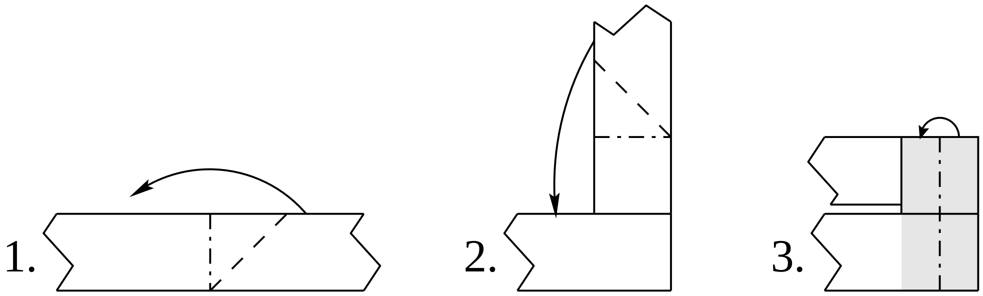


  - Assume by symmetry that we have a left turn
  - Fold behind, perpendicular to right edge at the turning point
  - Fold bottom layer back to make desired turn
  - Wrap excess paper around the corner

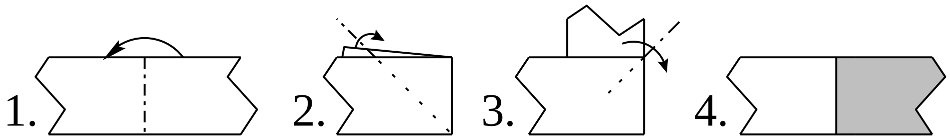
- Can construct an arbitrary strip ring by putting these turn gadgets together

Bridges

- Join two possibly overlapping 90-degree turn gadgets
  - Fold excess paper down to a thin rectangle

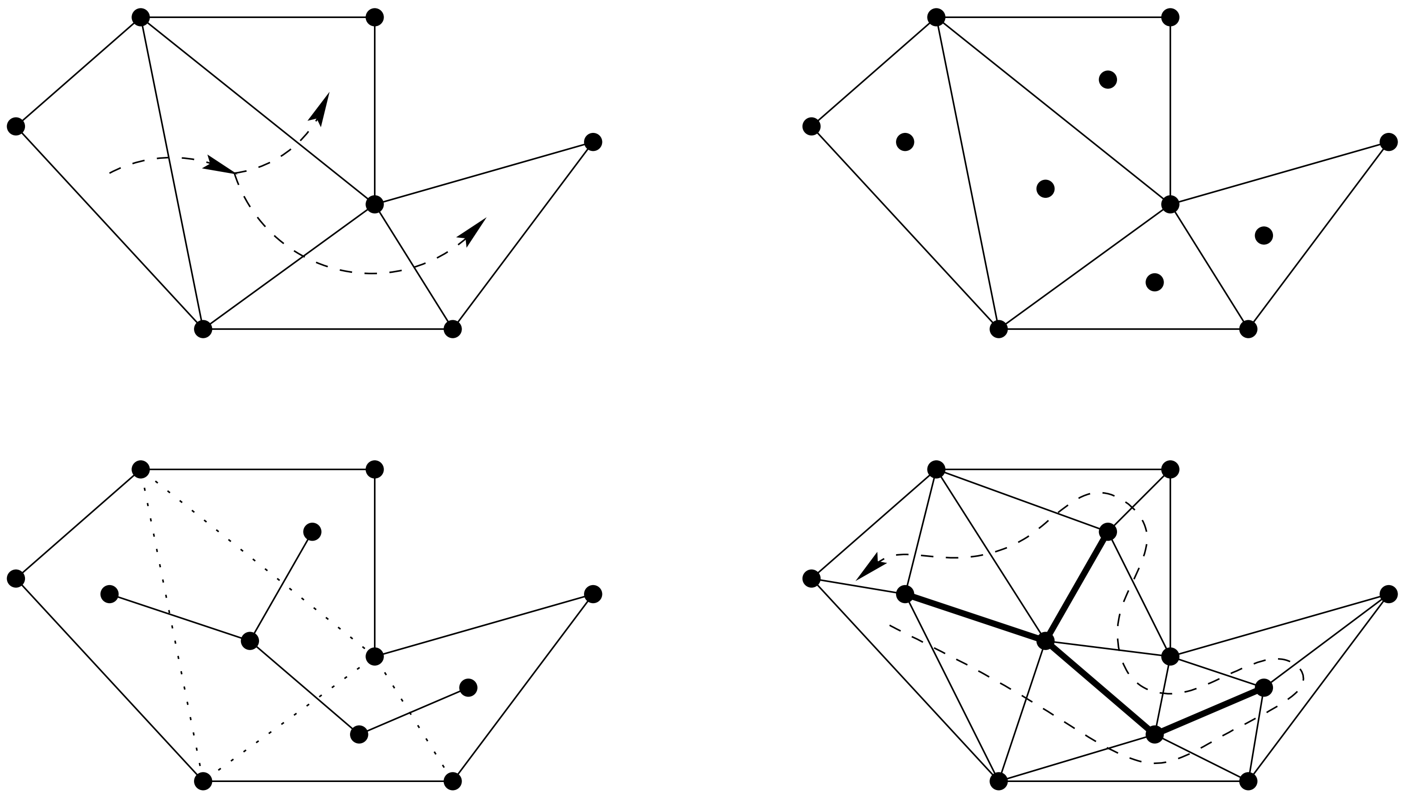


- Can find a place to put this thin rectangle along any edge of joining
- Optionally apply a color reversal gadget in between the two turns:



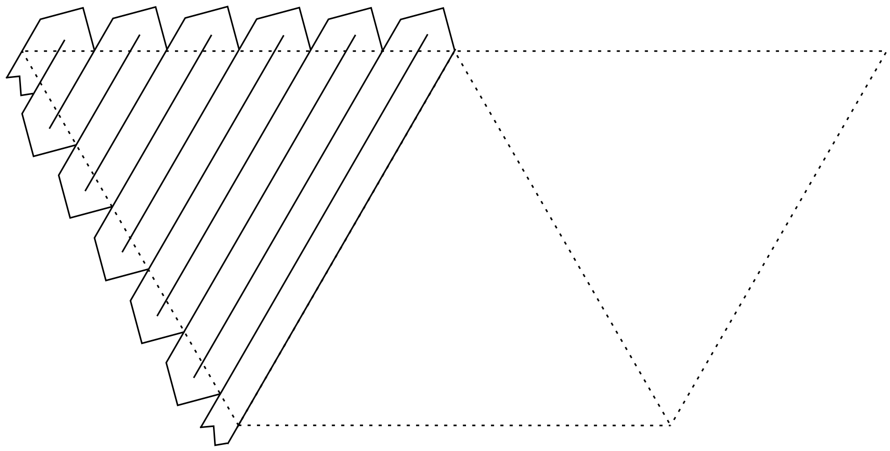
Triangle Approach

- **Hamiltonian triangulation**:
  There is a path visiting each triangle precisely once
- Arkin, Held, Mitchell, and Skiena 1996:
  In linear time can find $O(n)$ points to add,
  and find a Hamiltonian triangulation of the result



Triangle Approach

- Cover each triangle in turn along the Hamiltonian path



  - Start at vertex opposite next triangle
  - Zig-zag to cover the triangle
  - Choose parity so that you end up at the desired next vertex
  - Fold excess over

- For any $\epsilon > 0$, there is a rectangle with area $A + \epsilon$ that can construct a given polygonal region

Conclusion

- Presented two main methods for
  - Constructing a desired silhouette
  - Constructing a desired 2-color pattern
  - Wrapping a polyhedron
- Open problems:
  - Non-strip solution?
  - Number of folds or layers bounded in terms of number of bits to encode the input
  - Adaptation of strip width