Unfoldable Polyhedra with Triangular Faces

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Abstract. We present a triangulated closed polyhedron that has no edge unfolding, and a triangulated open polyhedron that has no unfolding whatsoever.

1 Introduction. A classic open question in geometry [3, 5] is whether every convex polyhedron can be cut along its edges and flattened into a single piece in the plane without any overlap. Such a collection of cuts is called an *edge unfolding* of the polyhedron, and the resulting simple polygon is called a *net.* It is widely conjectured that the answer to this question is yes. Currently it is only known that if we allow cuts across the faces as well as along the edges, then every convex polyhedron has an unfolding [5].

While unfoldings were originally used to make paper models of polyhedra, unfoldings have important industrial applications such as sheet metal bending [4, 5]. For these applications, nonconvex polyhedra are of the most interest. Surprisingly, there has been little theoretical work on unfolding nonconvex polyhedra. In what may be the only paper on this subject, Biedl et al. [2] show the positive result that certain classes of orthogonal polyhedra can be unfolded. They also show the negative result that not all orthogonal polyhedra have edge unfoldings.

However, their examples are not entirely satisfying because they are not even "topologically convex." A polyhedron is *topologically convex* if its graph (1-skeleton) is the graph of some convex polyhedron. Steinitz's theorem tells us that this is the case precisely if the graph is 3-connected and planar.

The class of topologically convex polyhedra includes all *convex-faced* polyhedra (i.e., polyhedra whose faces are all convex) that are homeomorphic

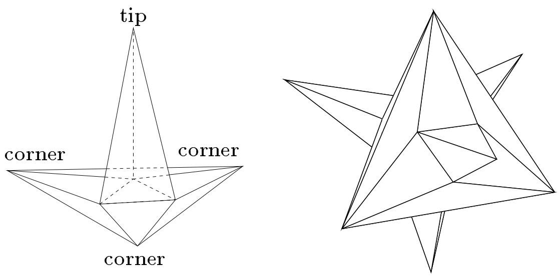

Figure 1: Triangulated hat (left) placed on each face of a regular tetrahedron (right). The right polyhedron is called the *spiked tetrahedron.*

to spheres. Schevon and other researchers [2, 6] have asked whether all such polyhedra can be unfolded by cutting along edges. In other words, can the conjecture that every convex polyhedron is edge-unfoldable be extended to topologically convex polyhedra?

Another particularly interesting subclass, which we consider here, are *triangulated* (or *simplicial*) polyhedra that are homeomorphic to spheres. In this paper, we give such a polyhedron that has no edge unfolding. This proves, in particular, that the edge-unfolding conjecture does not generalize to topologically convex polyhedra.

A preliminary version of this work [1] proves several lemmas used here, and presents a convex-faced (but not triangulated) polyhedron with no edge unfolding.

2 No Edge Unfolding. Our construction, the *spiked tetrahedron*, is as follows. On each face of a regular tetrahedron, erect a *triangulated hat* consisting of nine triangular faces as shown in Figure 1. The center *spike* has three isosceles triangles with base angle strictly greater than 60°. The remaining six triangles (called the *brim*) are nearly flat; they may be given convex dihedral angles by pushing the spike slightly above the plane of the tetrahedron's face.

The three outer vertices are the *corners* of the hat (which coincide with the tetrahedron's vertices), the innermost vertex is the *tip*, and the rest are the *middle* vertices. The edges that do not join corners are said to be *inside the hat.*

Lemma 1 *In any edge unfolding of the spiked tetrahedron, there is a path joining at least two corners of each hat using edges inside that hat (for suitable choices of the parameters).*

Proof: Consider an unfolding of the polyhedron, which must be a spanning tree of the vertices [1], and focus attention to the subforest of edges inside

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a particular hat. Suppose that this forest touches only one corner of that hat. Because the three middle vertices have negative curvature (i.e., the incident angles sum to more than 360°), they cannot be leaves of the unfolding [1]. Hence, the corner and tip must be the two leaves of the forest (which in fact must be a path), and only one cut is incident to a corner. Because the spike remains connected to the rest of the polyhedron, there must be a spike triangle A that remains connected to a brim triangle B; see Figure 2. Because there is only one cut incident to a corner, we can go either clockwise or counterclockwise half way around the hat to find brim triangles B, C, and D all sharing a vertex with A. But by making the brim so close to flat that the sum of angles from A, B, and C is closer to 300° than the base angle of A is to 60°, the sum of the angles of the four faces at that vertex is more than 360°, so A overlaps D when unfolded. □

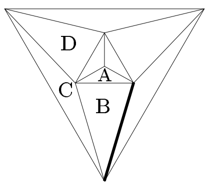

Figure 2: Proof of Lemma 1: a planar drawing of a triangulated hat with a possible cut in bold.

Theorem 1 *The spiked tetrahedron cannot be edge unfolded (for suitable choices of the parameters).*

Proof: Suppose there were a spanning tree that produced an edge unfolding. By Lemma 1, inside each of the four hats would be paths joining two corners. But since there are only four corners in total, these paths would form a cycle in the spanning tree, a contradiction. □

3 General Unfolding. The full version of this paper (manuscript in preparation) proves that the spiked tetrahedron always has a *general unfolding*, i.e., an unfolding in which cuts are allowed along faces in addition to edges of the polyhedron. A "picture proof" is given in Figure 3.

4 No General Unfolding. An intriguing open question is whether there is a convex-faced polyhedron, triangulated polyhedron, or any polyhedron that has no general unfolding. This section makes a step toward solving this problem by presenting a triangulated *open* polyhedron (polyhedron with boundary) with no general unfolding.

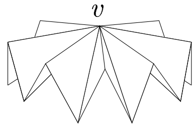

Figure 4: Open polyhedron with no unfolding.

The construction is to connect several triangles in a cycle, all sharing a common vertex v, as shown in Figure 4. By connecting enough triangles and/or adjusting the triangles to have large enough angle incident to v, we can arrange for vertex v to have negative curvature.

Theorem 2 [1] *The open polyhedron in Figure 4 has no general unfolding if v has negative curvature.*

Proof: An unfolding could only have leaves on the boundary, because v has negative curvature, and the cut incident to any other leaf could be glued (uncut) without affecting the unfolding. But any unfolding has at least two leaves, so must disconnect the polyhedron, contradicting the definition of unfolding. □

5 Conclusion. Our first example (Figure 1) shows that the conjecture about edge unfoldings of convex polyhedra cannot be extended to topologically convex polyhedra. It further illustrates the added power of cuts along faces for topologically convex polyhedra. Our second example (Figure 4) shows an open polyhedron that cannot be unfolded at all, but the flexibility exploited in Figure 3 suggests that it will be more difficult to settle whether there is a closed polyhedron with that property. Another interesting open question is the complexity of deciding whether a given triangulated polyhedron has an edge unfolding, now that we know that the answer is not always "yes."

Acknowledgments. We thank Anna Lubiw for helpful discussions.

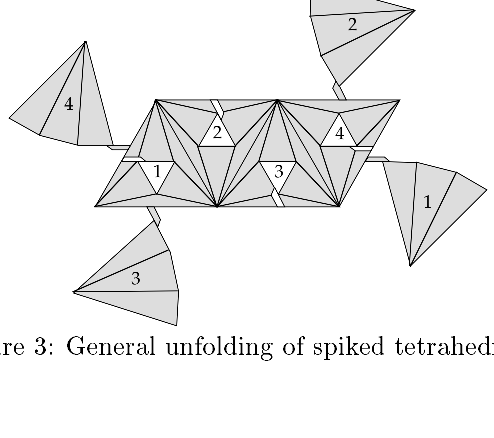

Figure 3: General unfolding of spiked tetrahedron.

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