When Can You Fold a Map?

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The easiest way to refold a road map is differently.
— Jones's Rule of the Road (M. Gardner [3])

1 Introduction

We explore the following class of problems: given a collection of creases on a piece of paper, each assigned a folding direction of mountain or valley, is there a flat folding by a sequence of *simple folds*? There are several models of simple folds. An *all-layers simple fold* corresponds to how we fold maps: fold the entire object along a line segment. A *one-layer simple fold* can fold just a single layer of paper at once. A *some-layers simple fold* is the most general, allowing any number of layers to be folded.

It is known that the general problem of deciding flat foldability of a crease pattern is NP-hard (Bern and Hayes [2], SODA'96). In this paper we consider the analogous question for simple folds. Specifically, suppose we are given a crease pattern (an embedded plane graph) and a specification of whether each crease is folded "mountain" or "valley." Can this *mountain-valley pattern* be folded flat by a sequence of simple folds? If so, we wish to construct such a sequence of folds.

Motivation. In addition to its inherent interest in the mathematics of origami, our study is motivated by applications in sheet metal and paper product manufacturing, where one is interested in determining if a given structure can be manufactured using a specified creasing machine (e.g., a "pan brake"), which is typically restricted to performing simple folds. While origamists can develop particular skill in performing non-simple folds to make beautiful artwork, practical problems of manufacturing with sheet goods require simple and constrained folding operations. Our goal is to develop a first suite of results that may be helpful towards a fuller understanding of the several manufacturing problems that arise, e.g., in making three-dimensional cardboard and sheet-metal structures.

Related Work. Our problem is related to the classic combinatorics question of *map folding* [3, 5]. This question asks for the *number* of foldings of a particular crease pattern, namely an $m \times n$ rectangular grid, by a sequence of simple folds. In contrast with this combinatorics question, we study the algorithmic complexity of the decision problem, also in some more general instances of crease patterns.

Our model of simple folding is also closely related to "pureland origami," a restriction introduced by Smith [6, 7]. Pureland folds include simple folds, but they also allow paper to be "tucked" into pockets, as well as "opened up" into three dimensions provided that no creases are made during the process.

Overview. The rest of this abstract summarizes our results. We first consider the analogous questions in one dimension lower: bending a segment into a flat object. These problems can be solved in essentially linear time. Then we explore various versions of simple foldability in two dimensions, and find a surprising contrast: map folding and variants are polynomial, but slight generalizations are NP-complete.

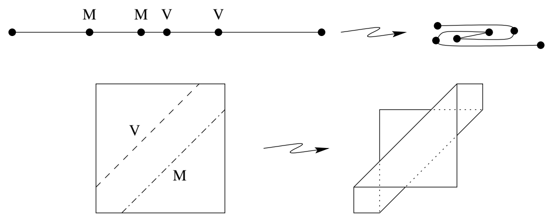

Figure 1: Sample flat foldings in one and two dimensions. Mountains and valleys are denoted by M's and V's, respectively.

2 1-D: One-Layer and Some-Layers

We prove the surprising result that a 1-D mountain-valley pattern is flat foldable precisely if it can be

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folded by a sequence of *crimps* and *end folds*, shown in Figure 2. In particular, a 1-D mountain-valley pattern is flat foldable precisely if it is simply foldable in the one-layer or some-layers model. These properties can also be tested in $O(n)$ time using basic data structuring.

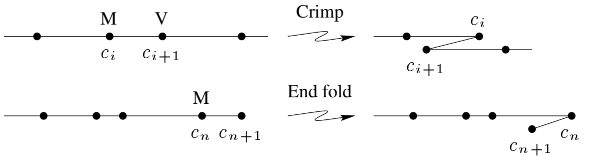

Figure 2: The two local operations for one-dimensional one-layer folds.

3 1-D: All-Layers Simple Folds

The 1-D all-layers simple-fold problem can be cast as a kind of "string folding" problem, by modeling a mountain-valley pattern as a string of M's and V's interspersed with distances. The basic goal is to find a crease to fold so that creases to the left and right match up complementarily; this corresponds to a substring matching the reverse complement of another substring. A key observation is that all folds made must have this property, and furthermore making such folds never destroys simple foldability, because making such a fold is equivalent to deleting a portion of the paper. We present two algorithms for finding valid folds and in the end testing simple foldability. One algorithm uses suffix trees augmented with a least-common-ancestor data structure, and runs in time linear in the bit complexity of the input if the distances are encoded in binary. Another algorithm uses randomized techniques from the Karp and Rabin string-matching algorithm [4] and achieves $O(n)$ expected time.

4 Orthogonal Simple Folds in 2-D

We show that simple foldability of a rectangular piece of paper with axis-parallel creases each prescribed either mountain or valley reduces to a sequence of 1-D problems with a few side constraints. This holds independent of the model of simple folds. Using some simple data structuring we obtain the same bounds as in 1-D: linear for one-layer and some-layers, linear in the bit complexity of the input for all-layers, and expected linear time for all-layers.

5 Hardness of Simple Folds in 2-D

We prove that the problem of deciding whether a 2-D axis-parallel crease pattern can be simply folded is (weakly) NP-hard, if we allow the initial paper to be an arbitrary rectilinear polygon. This result holds independent of the model of simple folds, and independent of whether the creases have a prescribed mountain-valley assignment. We also show that it is (weakly) NP-hard to decide whether a mountain-valley pattern on a square piece of paper can be folded by some-layers simple folds, if the creases are allowed to be axis-parallel and at a 45-degree angle. The reductions are from PARTITION.

Acknowledgments. We thank Jack Edmonds for helpful discussions which inspired this research. E. Arkin and J. Mitchell thank Mike Todd for posing algorithmic versions of the map-folder's problem in a geometry seminar at Cornell.

E. Arkin acknowledges support from the National Science Foundation (CCR-9732221) and HRL Laboratories. M. Bender acknowledges support from HRL Laboratories. J. Mitchell acknowledges support from HRL Laboratories, the National Science Foundation (CCR-9732221), NASA Ames Research Center, Northrop-Grumman Corporation, Sandia National Labs, Seagull Technology, and Sun Microsystems.

References

[1] E.M. Arkin, S.P. Fekete, J.S.B. Mitchell, and S.S. Skiena. On the manufacturability of paper-clips and sheet metal structures. Manuscript, 1999.

[2] M. Bern and B. Hayes. The complexity of flat origami. In *Proc. 7th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 175–183, 1996.

[3] M. Gardner. The combinatorics of paper folding. In *Wheels, Life and Other Mathematical Amusements*, Chapter 7, pages 60–73. W. H. Freeman, New York, NY, 1983.

[4] R.M. Karp and M.O. Rabin. Efficient randomized pattern-matching algorithms. *IBM Journal of Research and Development*, 31(2):249–260, March 1987.

[5] W.F. Lunnon. Multi-dimensional map-folding. *The Computer Journal*, 14(1):75–80, February 1971.

[6] J.S. Smith. Origami profiles. *British Origami*, 58, June 1976.

[7] J.S. Smith. *Pureland Origami 1, 2, and 3*. British Origami Society. Booklets 14, 29, and 43, 1980, 1988, and 1993.