Computational Geometry Column 37

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Abstract

Open problems from the 15th Annual ACM Symposium on Computational Geometry.

Following is a list of the problems presented on June 15, 1999 at the open-problem session of the 15th Symposium on Computational Geometry.

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Are there n points in $\mathbb{R}^3$ that have more than $n!$ different triangulated spheres?

An upper bound of $O(10^n n!)$ derives from Tutte's bounds on unlabeled planar triangulations times the ways to embed vertices, ignoring intersections. A lower bound of $\Omega((n/3)!)$ is established by a construction with $n/3$ thin needles.

Frequently in computer graphics, triangulated spheres are represented through two separate components: geometry (vertex coordinates) and topology (vertex connectivity). A negative answer to the posed question permits the compression of the topological information by storing it implicitly within the geometric information.

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Say that a nonconvex polyhedron in $\mathbb{R}^3$ is *smooth* if (a) every facet is a triangle with constant aspect ratio (i.e., the triangles are *fat*), and (b) the minimum dihedral angle (either internal or external) is a constant. Can any smooth polyhedron be triangulated (perhaps employing Steiner points) using only $O(n)$ tetrahedra? Equivalently, is it always possible to decompose a smooth polyhedron into $O(n)$ convex pieces?

Both conditions (a) and (b) are necessary. Chazelle's polyhedron [C84], which cannot be decomposed into fewer than $\Omega(n^2)$ convex pieces, has large dihedral angles, but its facets have aspect ratio $\Omega(n)$. A variant of Chazelle's polyhedron can be constructed with fat triangular facets, but with minimum dihedral angle $O(1/n^2)$.

Smooth polyhedra are not necessarily "fat"[1]—for example, a $1 \times n \times n$ rectangular "pizza box" can be approximated arbitrarily closely by a smooth polyhedron with $\Theta(n)$ facets. The Schönhardt polyhedron (a nontriangulatable twisted triangular prism [E97]) is smooth, so smooth polyhedra cannot necessarily be triangulated without Steiner points. In fact, gluing $O(n)$ Schönhardt polyhedra onto a sphere yields a smooth polyhedron that satisfies both conditions and requires $\Omega(n)$ Steiner points. Is this the worst possible?

[C84] B. Chazelle. Convex partitions of polyhedra: A lower bound and worst-case optimal algorithm. *SIAM J. Comput.* 13:488–507, 1984.

[E97] D. Eppstein. Three untetrahedralizable objects. Part of *The Geometry Junkyard.* May 1997. `http://www.ics.uci.edu/~eppstein/junkyard/untetra/`

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[1]Under any reasonable definition of *fat*, e.g., a bounded volume ratio of the smallest enclosing sphere to the largest enclosed sphere.

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What is the complexity of the Maximum TSP for Euclidean distances in the plane?

Barvinok et al. [BJWW98] have shown that the *Maximum TSP* (i.e., the problem of finding a traveling salesman tour of maximum length) can be solved in polynomial time, provided that distances are computed according to a polyhedral norm in $\mathbb{R}^d$, for some fixed d. The most natural instance of this class of problems arises for rectilinear distances in the plane $\mathbb{R}^2$, where the unit ball is a square. With the help of some additional improvements by Tamir, the method by Barvinok et al. yields an $O(n^2 \log n)$ algorithm for this case. This was improved in [F99] to an $O(n)$ algorithm that finds an optimal solution.

For the case of Euclidean distances in $\mathbb{R}^d$ for any fixed $d \geq 3$, it has been shown [F99] that the Maximum TSP is NP-hard. The case of $d = 2$ remains open, but the problem poser conjectures it to be NP-hard.

[**BJWW98**] A. I. Barvinok, D. S. Johnson, G. J. Woeginger, and R. Woodroofe. The maximum traveling salesman problem under polyhedral norms. *Proc. 6th Internat. Integer Program. Combin. Optim. Conf.*, Springer-Verlag, Lecture Notes in Comput. Sci. 1412, 1998, 195–201.

[**F99**] S. P. Fekete. Simplicity and hardness of the Maximum Traveling Salesman Problem under geometric distances. *Proc. 10th Annu. ACM-SIAM Sympos. Discrete Algorithms*, 1999, 337–345.

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The "wireless communications location problem" is to determine the point p on the $z = 0$ plane at which a user is located from signals detected by a collection of receivers. The user sends out a radio signal (a sphere expanding at uniform velocity), which perhaps diffracts around or reflects off of buildings before being received. The given data is as follows:

- A set B of orthogonal 3D boxes (the buildings), oriented isothetically, each with its base on the $z = 0$ plane, each taller than the highest receiver;
- A set R of receiver locations (points in $\mathbb{R}^3$); and
- A set S of received signals, each described by the receiver and the distance traveled along the signal's polygonal path.

The signals S come in three varieties:

1. Line-of-sight (LOS) signals, direct from p to $r \in R$.
2. 1-diffracted signals, diffracted at most once around an edge or vertex of some building $b \in B$. In effect the expanding sphere of radio waves is retransmitted at building corners.
3. k-reflected signals, reflected at most k times off of building walls (which act as a radio-wave mirrors). It may be assumed that $k \leq 3$.

LOS signals can be distinguished from diffracted and reflected signals. It may be assumed that no signal is both diffracted and reflected. Various pragmatic assumptions may be made about the maximum propagation distance of a signal, and the size of the buildings with respect to this distance; see [PS99] and [F96] for the technical details. An approximate location for p with bounded error would suffice. A natural question is determining necessary conditions for accurate location of the user. Preprocessing can be as extensive as needed.

[**PS99**] G. Perkins and D. Souvaine. Efficient radio wave front propagation for wireless radio signal prediction. DIMACS Tech. Rep. 99-27, May 1999.
`http://dimacs.rutgers.edu/TechnicalReports/1999.html`

[**F96**] S. Fortune. A beam-tracing algorithm for prediction of indoor radio propagation. *Proc. 1st ACM Workshop on Appl. Comput. Geom.* Eds. M. Lin and D. Manocha. Springer-Verlag, Lecture Notes in Comput. Sci. 1148, 1996, 76–81.

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Define a *prism* $\mathcal{P}$ as a polyhedron formed by extruding a polygon P in the $z = 0$ plane to the $z = 1$ plane. Call the bottom and top faces P_0 and P_1 respectively. Given two different triangulations of P_0 and P_1, each perhaps using Steiner points, determine whether $\mathcal{P}$ can be tetrahedralized respecting the triangulations of P_0 and P_1, perhaps with Steiner points, but using only $O(n)$ tetrahedra. If so, provide an algorithm to find such a tetrahedralization.

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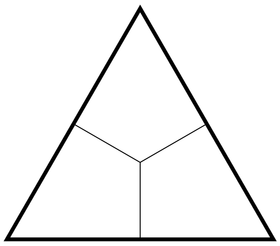
Is there a topological cube with orthogonal opposite facets? More precisely, does there exist a bounded convex polytope in $\mathbb{R}^3$, whose graph (1-skeleton) is the same as that of a unit cube (6 quadrangular facets with the incidence relationships of a cube), such that for each of the three pairs of opposite facets, the planes containing them form a dihedral angle of 90 degrees?

During the open-problem session, John Conway observed that there is a nonconvex topological cube with this property. His construction is as follows. Take three quadrangles arranged in the plane as shown to the right, and add three vertical rectangular faces at the bold edges, meeting at a vertex at infinity. Then "distort" this polyhedron slightly to make a topological cube with orthogonal opposite faces. The center vertex dents inwards, so the polyhedron is nonconvex.



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Solve either of the following problems in $o(n^4)$ time:

1. Let S be a set of n segments in the plane. Determine whether there is a point p seeing all segments completely.
2. Let P be a set of n points in the plane. Find a point q in the plane that maximizes the minimum difference between any pair of distances $d(p, q)$ and $d(r, q)$ over all pairs $p, r \in P$. That is, find a point q that maximizes $\min_{p,r\in P} |d(p, q) - d(r, q)|$.

The goal of the second problem is to reduce the dimension of P from two to one by replacing every point by its distance to the point q. For this to be effective, q should be chosen so that it discriminates between any pair of points as much as possible.

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Let P be a set of n points in the plane. Call a unit circle *good* if it touches at least three points from P. What is the maximum number of good unit circles that can be placed, maximized over all possible sets of n points?

All that is known is a trivial upper bound of $O(n^2)$, and a lower bound of $2n$ (on a lattice). This problem was mentioned in [B83]; nothing more recent is known. It has an application in estimating the complexity of certain pattern-matching problems.

[B83] J. Beck. On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica* 3(3–4): 281–297, 1983.

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Several classic problems were reposed:

Angel Problem: Consider the following game played on an infinite chessboard. In each move, the *angel* can fly to any uneaten square within 1,000 king's moves; and the *devil* can eat any one square (while the angel is aloft). The angel wins if it has a strategy so that it can always move; otherwise the devil wins. Prove who wins. *Reward*: \$1,000 if the devil wins; \$100 if the angel wins.

[BCG82] E. R. Berlekamp, J. H. Conway, R. K. Guy. *Winning Ways for Your Mathematical Plays.* Volume 2: *Games in Particular.* Academic Press, 1982, p. 609.

Thrackle Problem: *Thrackles* are made of "spots" (points in $\mathbb{R}^2$) and "paths" (smooth closed curves, ending at spots), with the condition that any two paths intersect at exactly one point, and have distinct tangents at that point. Can there be more paths than spots? *Reward*: \$1,000.

[**O95**] J. O'Rourke. Computational geometry column 26. *Internat. J. Comput. Geom. Appl.* 5:339–341, 1995. Also in *SIGACT News* 26(2):15–17, 1995.

Holyhedron Problem: Is there a polyhedron with a hole in every face (i.e., every face of which is multiply connected)? *Reward*: \$10,000 / number of faces in the discovered polyhedron. The reason for the divisor in the reward is that a solution is known for an astronomical number of faces.

Danzer's Problem: Given an infinite set S of points in $\mathbb{R}^2$, with the property that there are at most k points of S in each disk of radius 1, must there be arbitrarily large convex "holes," i.e., regions containing no points of S?

[**CFG91**] H. P. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, 1991. Problem E14.