Folding a Better Checkerboard

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Abstract. Folding an $n \times n$ checkerboard pattern from a square of paper that is white on one side and black on the other has been thought for several years to require a paper square of semiperimeter n^2. Indeed, within a restricted class of foldings that match all previous origami models of this flavor, one can prove a lower bound of n^2 (though a matching upper bound was not known). We show how to break through this barrier and fold an $n \times n$ checkerboard from a paper square of semiperimeter $\frac{1}{2}n^2 + O(n)$. In particular, our construction strictly beats semiperimeter n^2 for (even) $n > 16$, and for $n = 8$, we improve on the best seamless folding.

1 Introduction

Within the world of origami, the use of two-colored paper (white on one side, colored on the other) has been widespread for many years, leading to creation of origami figures in which both colors are used for artistic effect. In the early days of western origami, two-colored figures tended to have relatively simple patterns (the penguin [11] has been a perennial favorite), but with the 1993 publication of John Montroll's *Origami Inside-Out* [10], the genre received a significant boost, as this book displayed much more complex patterns: striped tigers, spotted cows, and most notably, an 8×8 checkerboard with squares alternating in color—each folded from a single square sheet with no cuts.

The checkerboard, in fact, has a long history as an origami subject. The smaller sizes—2×2 and 3×3—make interesting puzzles, accessible even to relative beginners at folding. Larger checkerboards, and in particular the 8×8 used for chess, are significant challenges to the origami designer. Even so, there are several solutions in the origami literature. To the best of our knowledge, the earliest chessboard from a single uncut square was folded by Max Hulme [7] in 1977, but several other designs exist by now [1, 8, 2, 5].

One of the most important attributes of complex origami designs such as these is their *efficiency*, which is simply a measure of the size of the finished

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figure relative to the original sheet of paper. For the design of a checkerboard with unit squares, a measure of the efficiency is the size, in the same units, of the square (or other shape) of paper from which it is folded. For example, among 8×8 boards, Hulme's [7] is folded from a 64×64 square, Casey's [1] and Kirschenbaum's [8] are folded from 40×40 squares, Montroll's [10] is folded from a 36×36 square, and Dureisseix's [5] and Chen's [2] are folded from 32×32 squares.

A secondary question, which plays more of an aesthetic role, is whether any square of the folded board is crossed by a folded edge. The most aesthetically pleasing checkerboards have *seamless* squares, in which each square is an unbroken surface. If one or more squares is crossed by a folded edge, we call such a solution *unconstrained*. The most common form of unconstrained square has a folded edge crossing its diagonal.

We may then ask a general question: what is the smallest square of paper required to fold an $n \times n$ checkerboard with unconstrained or seamless squares?

Origamists have experimented with this problem for specific values of n other than just 8. Figure 1 shows sample foldings for $n \in \{2, 3, 4\}$. The 2×2 and 3×3 solutions are elegant and have been plausibly conjectured, although not yet proved, to be optimal. Foldings become increasingly difficult as n grows. Although the three examples in Figure 1 are all seamless, in the larger sizes both seamless and unconstrained square solutions exist, with unconstrained solutions often turning out to be slightly more efficient. In particular, an unconstrained 4×4 checkerboard can be folded from an 8×8 square [5, 7].

Perimeter limit. An interesting property of many examples is that the perimeter of the square in the folded form follows the edges of the colored/white boundary (or exterior boundary) in the pattern. Indeed, finding a way to map the boundary of the square to the colored/white boundary of the surface pattern has been a powerful and fruitful design approach for checkerboards (as well as many other two-colored origami figures). However, in many of the folded examples not all of the boundary of the square is used to create colored/white boundary in the pattern; there are often small bits of boundary that double back on themselves or that remain hidden from view as part of the folding design. The significance of the square perimeter in the construction of checkerboards was first explicitly identified by Dureisseix [5] in his construction of an unconstrained-square 8×8 from a 32×32 square. See also [4, sec. 15.4.2, pp. 238–239] for a discussion.

For an $n \times n$ checkerboard of unit squares viewed as a pattern of white squares on a colored field, the total boundary of the $n^2/2$ white squares (both white/colored and white/boundary) is $4(n^2/2) = 2n^2$, which leads directly to the following conjectured relationship between the size of a checkerboard and the minimum size square (or, in general, any other convex shape) required to fold it:

Conjecture 1 (Checkerboard Perimeter Limit). The minimum scaling of a convex polygon required to fold an $n \times n$ checkerboard of unit squares has semiperimeter at least n^2.

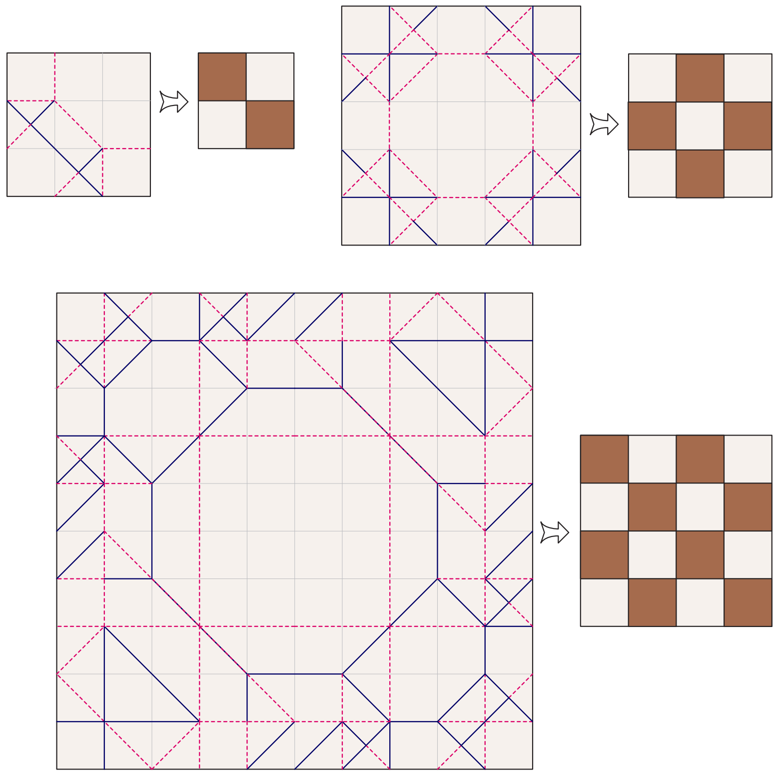

Fig. 1: Top left: Seamless 2×2 checkerboard from a 3×3 square, crease pattern (left) and folded form (right). Top right: Seamless 3×3 checkerboard from a 5×5 square. Bottom: Seamless 4×4 checkerboard from a 10×10 square. Dashed lines are valley folds; solid lines are mountain folds.

For folding from a square, the minimum size square would have a side length of $L_{\min} = \lceil n^2/2 \rceil$. Thus, for example, the paper required to fold an 8×8 checkerboard would have a semiperimeter of 64, and so the minimum size square would be at least 32×32. This conjectured limit would seem to be borne out by the published examples of 8×8 checkerboards. In particular the designs of Chen and Dureisseix achieve this bound. Also, while Montroll's published board is folded from a 36×36 grid, he has reported in personal communication with the authors that he has an unpublished design for a checkerboard from a 32×32 grid. On the other hand, the best known seamless 8×8 boards [1, 8] use 40×40 unit squares.

Table 1 lists the conjectured lower bounds for n from 2 to 8 for square paper, along with the best known unconstrained and seamless solutions.

Our results. After a certain amount of experimentation, we found that these limits seemed plausible, and a challenge to even meet. To our knowledge, the

n	conjectured limit	best previous example	
		unconstrained	seamless
2	2	3	3
3	5	5	5
4	8	8	10
5	13	?	?
6	18	?	?
7	25	?	?
8	32	32 [2, 5]	40 [1, 8]

Table 1: Checkerboard size, conjectured limit on square size, and sizes of best previous known unconstrained and seamless solutions.

only general construction for $n \times n$ checkerboards for arbitrary n is what follows from a general construction of two-color patterns [3]. This construction starts by accordion-folding a paper square into a long rectangular strip, and "flips" the strip at each color reversal. This method requires a square of side length $2n^2 + O(n)$, or semiperimeter $4n^2 + O(n)$, though it is at least seamless.

An improvement we developed, shown in Figure 2, asymptotically approaches the conjectured lower bound for unconstrained squares. In this approach, the perimeter of the checkerboard is mapped precisely to the checkerboard, with a bit effectively "wasted" in the turns at the end. However, this excess material increases only linearly with n. For an $n \times n$ checkerboard (n even), the amount of "square diagonal" needed, including turns, can be shown to be $\frac{1}{2}n^2 + 4n - 5$, which leads to a semiperimeter of

$$n^2 + 8n - 10,$$

which, indeed, asymptotically approaches the limit of n^2 for large n (though exceeds n^2 for all $n \geq 2$). To our knowledge, this is the first construction for general n other than the straightforward folding from a rectangular strip that follows from a general construction of two-color patterns [3].

Based on these examples, algorithms, and many empirical folding tests, the perimeter conjecture seemed to all of the present authors likely to be the true limit for an $n \times n$ checkerboard. Thus, it was with some surprise that we discovered a new approach to folding a checkerboard that beats the conjectured limit, at least for sufficiently large n. This paper presents an algorithmic implementation of this approach and shows that it leads to a semiperimeter asymptotically shorter than the conjectured limit by a factor of 2:

Theorem 1. *A seamless $n \times n$ checkerboard can be folded from a square of semiperimeter $\frac{1}{2}n^2 + O(n)$.*

Our construction strictly beats semiperimeter n^2 for all (even) $n > 16$. Along the way, we present a seamless checkerboard construction from a rectangle of semiperimeter $\frac{1}{2}n^2 + O(n)$ but aspect ratio $\Theta(n)$. This construction achieves better lower-order terms, in particular improving the best seamless 8×8 checkerboard folding from 40×40 to $34 + \varepsilon \times 34 + \varepsilon$ for any $\varepsilon > 0$.

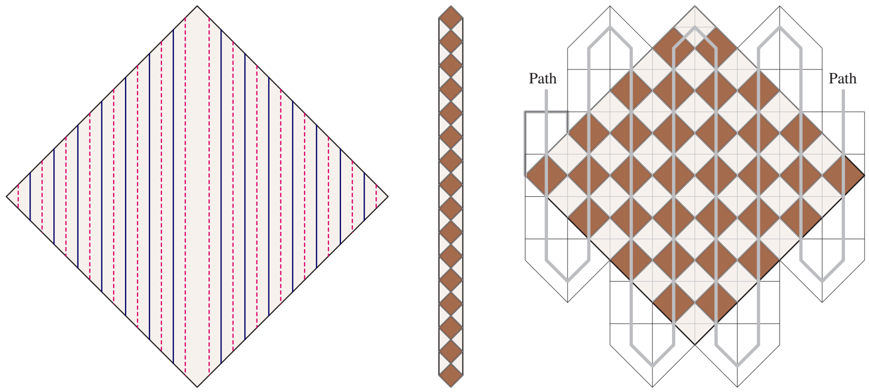

Fig. 2: Top: A square is pleated diagonally to realize a strip of squares and half-squares. Bottom: The diagonal is wound back and forth across the desired checkerboard, with folded turns at the ends.

Finally, our approach suggests a general technique for folding an arbitrary two-color pattern. This technique is based on solving a TSP-with-neighborhoods instance, which has the potential to beat the natural limit determined by perimeter length (an Euler tour). However, fully optimizing the construction does not appear to be easy, and the TSP-with-neighborhoods solution provides only a lower bound for the construction.

2 Construction

In the full paper, we give two efficient foldings of a checkerboard from paper of semiperimeter $\frac{1}{2}n^2 + O(n)$. The first construction uses a rectangle of paper of aspect ratio $\Theta(n)$. The second construction is from a square of paper. We focus here on details of the square construction, but touch on the rectangle construction to build intuition and for its smaller lower-order terms.

2.1 Rectangular Paper

Figure 3 shows the idea of our efficient checkerboard folding from a rectangular piece of paper. We fold the paper into an $n \times n$ square with n square tabs sticking up in alternate rows, and with strips of length $n/2$ hanging off the sides of each row. The back sides of the strips are colored, while the $n \times n$ square and tabs are white. Thus folding the strips to cover the $n \times n$ square turns it colored, and folding the white tabs alternating up and down makes a checkerboard pattern.

The effect of this approach is that, instead of tracing the boundary of the white (or colored) regions with the paper perimeter, we use the paper perimeter to create strips of color that can visit all the squares in the checkerboard. The tabs can be constructed from the pleats introduced by strip folding, without increasing the demand on perimeter.

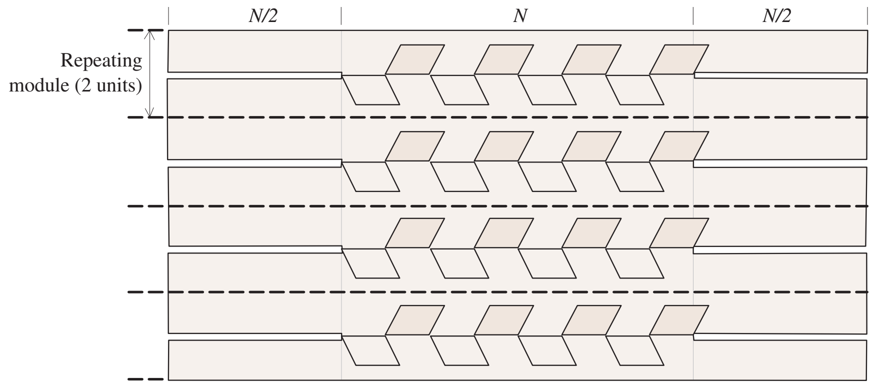

Fig. 3: Efficient construction of a checkerboard from rectangular paper.

The folding uses a $\frac{1}{2}n^2 + 2 + \varepsilon \times 4n + \varepsilon$ rectangle, for a semiperimeter of $\frac{1}{2}n^2 + 4n + 2 + \varepsilon$ for any desired $\varepsilon > 0$. This bound beats the conjectured limit for all $n > 8$. An 8×8 checkerboard would use a $34 + \varepsilon \times 32 + \varepsilon$ rectangle, better than the 40×40 of the best previous seamless checkerboards [1, 8] and respectably close to the 32×32 of the best unconstrained checkerboards [2, 5].

2.2 Square Paper

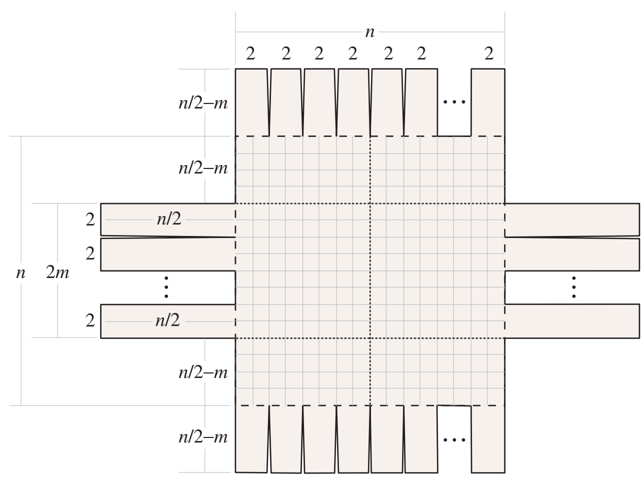

Fig. 4: Dissection pattern for an $n \times n$ square (n even).

Our construction from a square piece of paper uses the same overall approach, but rebalanced using a different arrangement of strips shown in Figure 4. The arrangement is parameterized by an integer m which we will set to minimize the aspect ratio of the paper. Along the left and right, we place m strips of width 2 and length $n/2$, which folded inward cover the middle of the square from left to right. Along the top and bottom, we place $n/2$ strips of width 2 and length $n/2 - m$, which folded inward cover the remaining area.

Figure 5 shows the necessary gadgets to fold this pattern of strips. All pleats fall on the half-integer grid. Each depth-L slot between two strips consumes a perimeter segment of length $2L+2$. Turning a corner between a horizontal strip of length L_w and a vertical strip of length L_h happens at a corner of the paper, and consumes $L_w + L_h + 1$ of the vertical perimeter and L_h of the horizontal perimeter. (This corner gadget can be flipped to consume more horizontal than vertical perimeter, but we use only the one orientation for simplicity; tuning here would only improve by an additive constant.)

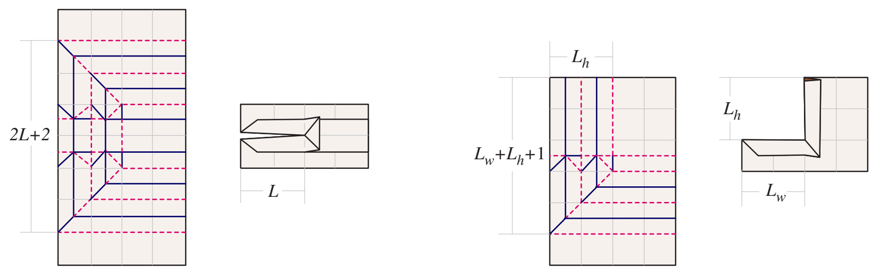

Fig. 5: Left: crease pattern and folded form for a slot (here with $L = 2$). Right: crease pattern and folded form for a corner (here with $L_w = L_h = 2$).

Although the construction is not yet complete, we compute the semiperimeter used so far. The vertical edge consists of m strips of width 2, $m-1$ slots of depth $L = n/2$, and two contributions from the upper and lower reflex corners with $L_w = n/2$ and $L_h = n - 2m$, giving a total of

$$V_1 = 2m + (m-1)\,(2(n/2)+2) + 2\,(n/2 + (n-2m) + 1) = mn + 2n.$$

The horizontal edge consists of $n/2$ strips of width 2, $n/2 - 1$ slots of depth $L = n/2 - m$, and two contributions from the left and right reflex corners with $L_w = n/2$ and $L_h = n - 2m$, giving a total of

$$H_1 = 2(n/2) + (n/2-1)\,(2\,(n/2-m)+2) + 2(n-2m) = \tfrac{1}{2}n^2 - mn + 3n - 2m - 2.$$

In the middle of the square, we fold an $n \times n$ array of *universal tabs* as shown in Figure 6. The universal tab creates a tab joined along a desired edge of the square that it covers, with the additional feature that the crease pattern's interface is identical on all four sides, consisting of four pleats, two on each side of the symmetry line, each pleat $\frac{1}{2}$ unit wide. Thus, a universal tab, rotated into any of the four possible orientations, can be folded from a 5×5 square (outlined by the heavy dashed line in Figure 6) whose creases will mate with the creases of adjacent universal tabs, no matter their orientations. Thus we can choose the orientation of each universal tab independently.

We orient universal tabs covered by horizontal strips to join along a horizontal edge (top or bottom), and we orient universal tabs covered by vertical strips to join along a vertical edge (left or right). This choice enables the strips to thread alternately above and below each universal tab it visits, forming the desired checkerboard color pattern.

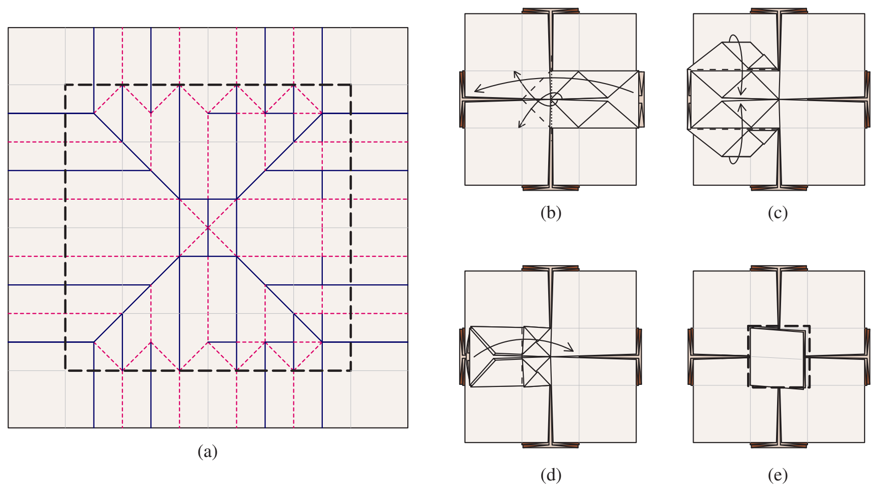

Fig. 6: (a) Crease pattern for the left-oriented universal tab. (b) Collapse the crease pattern to this, then spread two layers and fold the flap to the left. (c) Fold two layers to the center line. (d) Fold the flap back to the right. (e) The completed universal tab with hinge along the left side. The heavy dashed line in the crease pattern maps to the boundary of the tab in the last figure.

It may seem wasteful to fold n^2 universal tabs when we need only half that many, but the perimeter consumed by these tabs is only linear in n (in particular, at most $5n$). We also obtain the feature that the same model can form any $n \times n$ bitmap of colored and white pixels: without changing the crease pattern, and just changing the overlap order, the strips can go either over or under each universal tab, resulting in the two possible colors. This idea of a universal pixel display was introduced to us by Masashi Tamaka[4] who designed a 4×4 model.

We can optimize the construction by re-using the pleats from the slots and reflex corners in the folding of the tabs. Each slot creates at least two pleats on either side of the slot, with the important property that the pleats have the correct parity to join up with the pleats in the universal tabs. Figure 7 shows where additional pleats must be added; each heavy dashed line indicates a single pair of pleats at that location, contributing two units to the perimeter along that side. Looking first at the vertical direction, we see that each of the m strips requires four additional pleats running down the middle of the strip, adding $4m$ to the vertical dimension. There are also $4(n/2 - m)$ horizontal pleats in each of the sections above and below the horizontal strips, for a total of $V_2 = 2n$ units added to the vertical edge length. In the other dimension, we must add four vertical pleats running down the middle of each of the $n/2$ strips, plus two pleats along the left and right boundary, for a total of $H_2 = 2n + 2$ units added to the horizontal edge length.

[4] Personal communication, June 2009.

One loose end to tie up: each universal tab requires exactly four pleats in each direction, two on each side of the vertical and horizontal symmetry lines. For the sets of pleats that come from slots or reflex corners, there will in general be more than four pleats running in one, the other, or both directions. This issue does not create any problems: we can put into place all but two pleats, then overlay the universal tab crease pattern on the "sandwich" of layers to complete the tab.

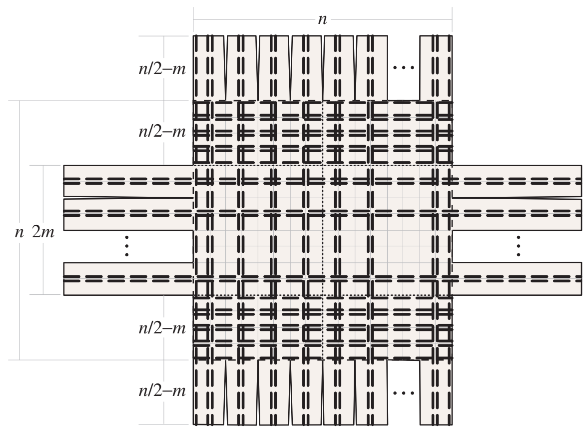

Fig. 7: Where pleats must be added to create universal tabs. Each heavy dashed line indicates a pair of pleats added at that location.

Therefore the entire crease pattern requires a rectangle of size

$$\begin{aligned} V &= V_1 + V_2 = mn + 2n + 4m \\ H &= H_1 + H_2 = \tfrac{1}{2}n^2 - mn + 3n - 2m - 2 + 2n + 2 \\ &= \tfrac{1}{2}n^2 - mn + 5n - 2m \end{aligned}$$

Setting $V = H$, we obtain

$$m = \frac{n(n+6)}{4(n+3)} = \tfrac{1}{4}n + \tfrac{3}{4} - O(1/n)$$

But m must be an integer, so we use either $m = \lfloor \frac{1}{4}(n+3) \rfloor$ or $m = \lceil \frac{1}{4}(n+3) \rceil$. In the first case, H dominates, while in the second case, V dominates. The best construction is the minimum of these two options, which (for n even) works out to a side length of

$$\tfrac{1}{4}n^2 + 4n + 4 - \tfrac{5}{2}(n \bmod 4).$$

The resulting semiperimeter is

$$\tfrac{1}{2}n^2 + 8n + 8 - 5(n \bmod 4),$$

which beats n^2 for all (even) $n > 16$. For $n = 8$, this construction uses a 52×52 square, worse than our rectangle construction, but for $n > 16$ the square construction wins.

3 More General Patterns

Our construction can be generalized to arbitrary two-color patterns. We fold the perimeter of the sheet into strips of lengths sufficient to reach every colored region of the final design. In case the pleats formed during the strip folding suffice to generate the tabs, we are done. Otherwise, we must make additional pleats first, before folding the strips.

There are multiple new ingredients that would be necessary for an optimal solution here. One issue is that these initial pleats and the strips together determine the width of the strips. Another is that with an irregular design, it may be necessary to run a complex optimization procedure to determine an optimal solution. A first attempt at formulating this optimization might be to notice that every colored region in the design must be reached by a strip, and therefore also by a point on the perimeter of the original sheet. Thus, in a solution, the perimeter will hit every colored region. In other words, the perimeter will form a solution to an instance of *TSP with neighborhoods* (also known as *one-of-a-set TSP* and *group TSP*).

However, finding such a tour will not be enough in general. Namely, colored regions of the folded design must be covered by the strips, not just touched by them. Thus larger colored regions in the design may require special wider strips, or multiple strips. We leave a more rigorous formulation of this optimization problem as a topic for future research.

Acknowledgments

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