Open Problems at the 2002 Dagstuhl Seminar on Algorithmic Combinatorial Game Theory

Erik D. Demaine

MIT Laboratory for Computer Science, Cambridge, MA 02139, USA
email: edemaine@mit.edu

Rudolf Fleischer

HKUST Department of Computer Science, Clear Water Bay, Kowloon, Hong Kong
email: rudolf@cs.ust.hk

Aviezri S. Fraenkel

Department of Computer Science and Applied Mathematics
Weizmann Institute of Science
Rehovot 76100, Israel
email: fraenkel@wisdom.weizmann.ac.il

Richard J. Nowakowski [1]

Department of Math. & Stats.
Dalhousie University
Halifax, NS B3H 3J5, Canada
email: rjn@mathstat.dal.ca

This is list of combinatorial games problems collected at the 2002 Dagstuhl Seminar on Algorithmic Combinatorial Game Theory, organized by the authors. For more unsolved problems in combinatorial games see the list of unsolved problems [4] and its successors in *Games of No Chance* [5] and in *More Games of No Chance* [7], hereafter referred to as MGONC. An on-line version can be found on David Wolfe's site [10]. Fraenkel's bibliography of papers about combinatorial games can be found in MGONC or on-line as a dynamic survey in the Electronic Journal of Combinatorics [2].

1. A **Subtraction game** is played with a heap of counters by two players moving alternately. There is a subtraction set $S = \{s_1, s_2, \ldots, s_k\}$ and on each

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turn a player may take away s_i counters, for some $i = 1, 2, \ldots, k$, from the heap. I. Althöfer and A. Flammenkamp propose a slight variant: the winner is the first player to make the heap size non-positive rather than the loser is the first player who cannot move where the heap size must be non-negative. A $\mathcal{P}-$position is one in where the **P**revious player has no good move and an $\mathcal{N}-$position one where the **N**ext player has a winning move. The end positions are $\mathcal{P}-$positions and recursively, a position is an $\mathcal{N}-$position if the player to move can move to a $\mathcal{P}-$position, otherwise it is an $\mathcal{N}$-position.

In either variant, the sequence of P and N positions must eventually be periodic since the determination of status of a heap of size n depends on only the k positions $n - s_i$, $i = 1, 2, \ldots, k$ and there are only 2^s possible sequences of length $s = \max\{s \in S\}$. Althöfer and Flammenkamp ask:

For a fixed $c > 0$ is it possible to find a subtraction game with maximum $s = \max\{s \in S\}$ and period length 2^{cs}?

The same question can be asked about *partizan* subtraction games, in which each player is assigned an individual subtraction set, see [3]. For subtraction sets S and T, $S > T$ if for all large enough n, S can always beat T. It is known that this order is non-transitive (i.e., there exists S, T and U with $S > T > U > S$). D. Wolfe asks:

Is there an efficient way of comparing subtraction sets?

[For more see Subtraction Games in [1, pp. 83–86] and [6].]

2: **Crash** is played on a strip of squares with heaps of red chips and heaps of blue counters [9]. The heaps are called towers. There are two players, Left and Right. Left, on her turn, chooses a heap of blue counters, picks up any number, say k (possibly all), of the counters and distributes then one per square moving to the right. If any of these squares contain red counters then these are removed from the board. Right on his turn, chooses a subset of some red heap and distributes them to the left and removes any blue counters from these squares. The loser is the first player to lose all their counters.

Juerg Nievergelt asks *for a solution* to this game. He remarks that the game with two towers of 3 each has been solved.

3: A dyadic box in d dimensions is of the form $\prod_{i=1}^{d} I_i$ where I_i is an interval $[a/2^s, (a+1)/2^s]$, $0 \leq a < 2^s$. J. Spencer asks

For $d-$dimensions, in how many ways can 2^n dyadic boxes of size $1/2^n$ be chosen so that the $d-$dimensional unit cube be packed?

In 2-dimensions, there are 12 dyadic boxes of size $1/2^2$, 4 $1 \times 1/2^2$ rectangles, 4 $1/2^2 \times 1$ rectangles and the 4 $1/2 \times 1/2$ squares and there are 7 ways of covering the square. Let the number of ways be $f_d(n)$. It is known that $f_2(n) \sim c\alpha^{2n}$. Is it true that $\lim_{n\to\infty} f_d(n)^{1/2^n} = \infty$?

4: Subset sums. Bob Hearn asks:

Given a set S of n positive real numbers is there a set T of m positive real numbers satisfying the following conditions?

(a) any subset sum of S can be represented as a subset sum of T; and

(b) the numbers in T grow exponentially, i.e., each member of T is strictly larger than the sum of all the smaller numbers in T.

Note that if S consists of integers then T can be the powers of 2, from 1 to $2^{\lfloor \log_2(\max s \in S) \rfloor}$

[Note: Part (a) was solved by Stefan Schwarz during the workshop.]

5: Given a directed graph with weighted edges and a distinguished vertex v. The first player chooses an edge directed out from v and thereafter the players, in turn, choose an edge directed out from the terminal vertex of the path. The game ends when a directed cycle is formed. The first player gets a payoff equal to the sum of the weights on the cycle. Her goal is to maximize this payoff and the second players wants to minimize it. Uri Zwick asks:

Can the optimal strategies be found in polynomial time?

[Polynomial in terms of the binary representation of the weights.]

In a related question, M. Müller notes that if the graph were the positions of a game given in a database then repeating a position would be a draw.

How quickly can one player show that the game is a draw when the second player thinks they have a win?

In the single player case, this is equivalent to finding a Hamiltonian cycle.

6: The **Dreidel** game is played by children at Hannuka. Each child starts with $\$n$. There is a four-sided dreidl (die) marked with N for NISHT (nothing); G for GANZ (everything); H for HALB (half); and S for SHTEL (put in).

VERSION 1: The rules of play are: (1) at the beginning everybody puts \$1 into the pot; (2) While there is money in the pot, each player takes a turn spinning the dreidl. If it comes up N pass; G, take the whole pot; H, take half the pot (round up if odd number is encountered); S, put in \$2. When there is nothing in the pot go back to (1). Play until one player has everyone's money.

VERSION 2: The game finishes when one child has run out of money, i.e., puts their last \$1 into the pot.

C. Banderier asks:

What is the expected length of the game?

He remarks that:

Conjecture 1: D. Zeilberger has a \$25 conjecture that if there are 2 children each with \$$n$ then the expected length is $O(n^2)$;

Conjecture 2: the time to the first ruin, with k players, is $O(n^2)$;

Conjecture 3: Let T be the time to the first ruin, with k players, then

$$\frac{E^2[T]}{V[T]} = \frac{2}{k+1}$$

7: Domineering is played on an $m \times n$ checkerboard. A domino covers exactly 2 squares of the board. Left places her dominoes vertically and Right horizontally. No dominoes may overlap. E. Berlekamp asks:

Is the hottest position in Domineering the 3×3 *square minus a corner?*

This has a value of $\pm 3/2$.

Domineering on $2 \times n$ and $3 \times n$ boards is relatively well understood but not for any wider boards. Berlekamp asks:

What is the maximum value of $(4 \times n) - 2(2 \times n)$*? Is it bounded above by 1?*

The difference is positive since a $4 \times n$ board reduces to 2 $2 \times n$ boards if Left refuses to place a domino across the center line.

8: H. Landman asks:

What is the right search technique for combinatorial games in practice?

We would want that at least each of the Left Stop, Right Stop, Mean Value, and Temperature of the games be computed by the search and accurately bounded during the search. We would also want that the search be maximally efficient (in the sense that alpha-beta is, of visiting as few nodes as possible). Note that we can get the stops by running two alpha-beta searches, one for Left starting and one for Right starting, but that these don't help too much with the temperature. Recent efforts in this direction have been by Müller [8], but they are partial steps and have not reached the full goal.

9: J. Trump asks about the *complexity of* **Go.** Robson has proved that Go is EXPTIME complete but this was under the simple Ko rules — a Ko cannot be taken back immediately. What is the complexity under the Super-Ko rule which bans any repetition of a board position? Note that Kos can make a game tree have exponential depth so the Super-Ko rule may have an effect.

10: Cash Nim is the game of Nim played with $1 coins. At the begining of the game, there are n coins distributed in k piles. The winner gets $\$10^n$, the loser gets to keep all the coins that he has removed from the heaps. The player who can win at normal nim [strategy: the winner plays so that she leaves the nim-sum or XOR of the heap sizes to be 0] will still want to win since this gives the largest payoff. M. Albert and R. J. Nowakowski ask

What is the maximum payoff the loser can guarantee himself?

Note that the game with heaps of 15, 12, 3 is a second player win but the first player can take 7 from the 15 heap and the only winning reply is to take 1 from the 12 heap.

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