Deflating The Pentagon

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Abstract. In this paper we consider deflations (inverse pocket flips) of n-gons for small n. We show that every pentagon can be deflated after finitely many deflations, and that any infinite deflation sequence of a pentagon results from deflating an induced quadrilateral on four of the vertices. We describe a family of hexagons that deflate infinitely for a specific deflation sequence, yet induce no infinitely deflating quadrilateral. We also review the known understanding of quadrilateral deflation.

1 Introduction

A *deflation* of a simple planar polygon is the operation of reflecting a subchain of the polygon through the line connecting its endpoints such that (1) the line intersects the polygon only at those two polygon vertices, (2) the resulting polygon is simple (does not self-intersect), and (3) the reflected subchain lies inside the hull of the resulting polygon. A polygon is *deflated* if it does not admit any deflations, i.e., every pair of polygon vertices either defines a line intersecting the polygon elsewhere or results in a nonsimple polygon after reflection.

Deflation is the inverse operation of pocket flipping. Given a nonconvex simple planar polygon, a *pocket* is a maximal connected region exterior to the polygon and interior to its convex hull. Such a pocket is bounded by one edge of the convex hull of the polygon, called the *pocket lid*, and a subchain of the polygon, called the *pocket subchain*. A *pocket flip* (or simply *flip*) is the operation of reflecting the pocket subchain through the line extending the pocket lid. The result is a new, simple polygon of larger area with the same edge lengths as the original polygon. A convex polygon has no pocket and hence admits no flip.

In 1935, Erdős conjectured that every nonconvex polygon convexifies after a finite number of flips [5]. Four years later, Nagy [2] claimed a proof of Erdős's conjecture. Recently, Demaine et al. [3, 4] uncovered a flaw in Nagy's argument, as well as other claimed proofs, but fortunately correct proofs remain.

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In the same spirit of finite flips, Wegner conjectured in 1993 that any polygon becomes deflated after a finite number of deflations [8]. Eight years later, Fevens et al. [6] disproved Wegner's conjecture by demonstrating a family of quadrilaterals that admit an infinite number of deflations. They left an open problem of characterizing which polygons deflate infinitely. Ballinger [1] closed the problem for quadrilaterals by proving that all infinitely deflating simple quadrilaterals are in the family defined by Fevens et al. [6].

This paper attempts to advance the understanding of deflating n-gons beyond $n = 4$. We prove that every pentagon admitting an infinite number of deflations induces an infinitely deflating quadrilateral on four of its vertices. Then we show our main result: unlike quadrilaterals, every pentagon can be deflated after finitely many (well-chosen) deflations. Finally, we construct a family of infinitely deflatable hexagons that induce no infinitely deflating quadrilateral; however, they deflate infinitely only according to a specific deflation sequence.

2 Definitions and Notation

Let $P = \langle v_0, v_1, \ldots, v_{n-1} \rangle$ be a polygon together with a clockwise ordering of its vertices. Let $P^k = \langle v_0^k, v_1^k, \ldots, v_{n-1}^k \rangle$ denote the polygon after k arbitrary deflations, and P^* denote the limit of P^k, when it exists, having vertices v_i^*. Thus, the initial polygon $P = P^0$. The *turn angle* of a vertex v_i is the signed angle $\theta \in (-180°, 180°]$ between the two vectors $v_i - v_{i-1}$ and $v_i - v_{i+1}$. A vertex of a polygon is *straight* if the angle between its incident edges is 180°, i.e., forming a turn angle of 0°. A *flat polygon* is a polygon with all its vertices collinear. A *hairpin* vertex v_i is a vertex whose incident edges overlap each other, i.e., forming a turn angle of 180°.[6] A polygon vertex is *sharpened* when its absolute turn angle decreases.

3 Deflation in General

In this section, we prove general properties about deflation of arbitrary simple polygons. Our first few lemmata are fairly straightforward, while the last lemma is quite intricate and central to our later arguments.

Lemma 1. *Deflation only sharpens angles.*

This result follows from an analogous result for pocket flips, which only flatten angles (see, e.g., [7]). For completeness, we provide a proof.

Proof. Consider the chain $v_i, v_{i+1}, \ldots, v_j$ that is to be deflated across line ℓ passing through v_i and v_j. The two vertices v_{i+1} and v_{i-1} are on different sides of ℓ. After deflating the chain $v_i, v_{i+1}, \ldots, v_j$, v_{i+1} is reflected across ℓ and its reflection is v'_{i+1}. Consider the two triangles $v_{i-1}v_iv_{i+1}$ and $v_{i-1}v_iv'_{i+1}$. The

[6] This terminology was introduced in [4] where it plays a role in pocket flips.

sides $v_i v_{i+1}$ and $v_i v'_{i+1}$ have the same length (deflation preserves edge lengths). Because v_{i+1} and v'_{i+1} have the same distance from ℓ, and v'_{i+1} is on the same side of ℓ as v_{i-1}, then the length of $v_{i-1}v'_{i+1}$ is less than the length of $v_{i-1}v_{i+1}$. This implies that the angle opposite edge $v_{i-1}v'_{i+1}$ is smaller than the angle opposite edge $v_{i-1}v_{i+1}$ (by Euclid's Propositions I.24 and I.25). Thus, the angle at vertex v_i sharpens. □

Corollary 1. *Any n-gon with no straight vertices will continue to have no straight vertices after deflation, even in an accumulation point P^*.*

Lemma 2. *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, the absolute turn angle $|\tau_i|$ at any vertex v_i has a (unique) limit $|\tau_i^*|$.*

Proof. By Lemma 1, $|\tau_i|$ never increases. Also, $|\tau_i|$ is bounded in the range $[0, 360°)$. Hence, $|\tau_i|$ has a limit $|\tau_i^*|$. □

Corollary 2. *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, v_i^* is a hairpin vertex in some accumulation point P^* if and only if v_i^* is a hairpin vertex in all accumulation points P^*.*

Lemma 3. *Any n-gon with n odd and having no straight vertices cannot flatten in an accumulation point of an infinite deflation sequence.*

Proof. Suppose for contradiction that there is a flat accumulation point. By Lemma 1, this limit has no straight vertices, so all vertices must be hairpins. Hence, the edges of the polygon alternate left and right. Because the edges form a closed cycle, when the first edge goes left, the last edge has to come back right in order to close the cycle. Hence, the number of edges of a flat polygon must be even. Therefore, any polygon with an odd number of vertices cannot flatten. □

Lemma 4. *For any infinite deflation sequence $P^0, P^1, P^2, \ldots$, there is a subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) that is the pocket chain of infinitely many deflations.*

Proof. Label each time t with (i, j) if the t-th deflation has pocket chain $v_i, v_{i+1}, \ldots, v_j$ (with $j - i \geq 2$). There are only finitely many labels, but infinitely many deflations, so some label must appear infinitely often. This label (i, j) corresponds to the desired subchain $v_i, v_{i+1}, \ldots, v_j$. □

We conclude this section with a challenging lemma showing that infinitely deflating pockets flatten:

Lemma 5. *Assume $P = P^0$ has no straight vertices. If P^* is an accumulation point of the infinite deflation sequence $P^0, P^1, P^2, \ldots$, and subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) is the pocket chain of infinitely many deflations, then $v_i^*, v_{i+1}^*, \ldots, v_j^*$ are collinear and $v_{i+1}^*, \ldots, v_{j-1}^*$ are hairpin vertices. Furthermore, if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_j^*, then v_j^* is a hairpin vertex; and if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_i^*, then v_i^* is a hairpin vertex. In particular, if $j - i = 2$, then either v_i^* or v_j^* is a hairpin vertex.*

Proof. Because $P^0 \supseteq P^1 \supseteq P^2 \supseteq \cdots$, we have $\mathrm{hull}(P^0) \supseteq \mathrm{hull}(P^1) \supseteq \mathrm{hull}(P^2) \supseteq \cdots$, and in particular $\mathrm{area}(\mathrm{hull}(P^0)) \geq \mathrm{area}(\mathrm{hull}(P^1)) \geq \mathrm{area}(\mathrm{hull}(P^2)) \geq \cdots \geq 0$. Thus, $\sum_{t=1}^{\infty}[\mathrm{area}(\mathrm{hull}(P^t)) - \mathrm{area}(\mathrm{hull}(P^{t-1}))] \leq \mathrm{area}(\mathrm{hull}(P^0))$, so $\mathrm{area}(\mathrm{hull}(P^t))$ $-$ $\mathrm{area}(\mathrm{hull}(P^{t-1})) \to 0$ as $t \to \infty$. Hence, for any $\epsilon > 0$, there is a time T_ϵ such that, for all $t \geq T_\epsilon$, $\mathrm{area}(\mathrm{hull}(P^t)) - \mathrm{area}(\mathrm{hull}(P^{t-1})) \leq \epsilon$. As a consequence, for all $t \geq T_\epsilon$, $\mathrm{hull}(P^{t-1}) \subseteq \mathrm{hull}(P^t) \oplus D_{\epsilon/\ell}$ where $\oplus$ denotes Minkowski sum, D_x denotes a disk of radius x, and ℓ is the length of the longest edge in P, which is a lower bound on the perimeter of $\mathrm{hull}(P^t)$.

Let $t_1, t_2, \ldots$ denote the infinite subsequence of deflations that use $v_i, v_{i+1}, \ldots, v_j$ as the pocket subchain, where P^{t_r} is the polygon immediately after the rth deflation of the pocket chain $v_i, v_{i+1}, \ldots, v_j$. Consider any vertex v_k with $i < k < j$. If $t_r \geq T_\epsilon$, then $v_k^{t_r-1} \in \mathrm{hull}(P^{t_r}) \oplus D_{\epsilon/\ell}$. Also, $v_k^{t_r-1}$ is in the halfplane H_r exterior to the line of support of P^{t_r} through $v_i^{t_r}$ and $v_j^{t_r}$. Now, the region $(\mathrm{hull}(P^{t_r}) \oplus D_{\epsilon/\ell}) \cap H_r$ converges to a subset of the line $\ell_{i,j}^{t_r}$ through $v_i^{t_r}$ and $v_j^{t_r}$ as $\epsilon \to 0$ while keeping $t_r \geq T_\epsilon$. Thus, for any accumulation point P^*, v_k^* is collinear with v_i^* and v_j^*, for all $i < k < j$. In other words, $v_{i+1}^*, \ldots, v_{j-1}^*$ lie on the line $\ell_{i,j}^*$ through v_i^* and v_j^*. By Corollary 1, $v_{i+1}^*, \ldots, v_{j-1}^*$ are not straight, so they must be hairpins.

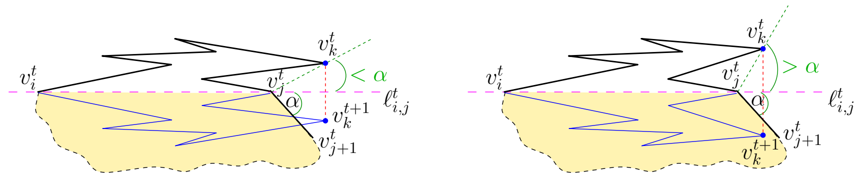
By Lemma 2, the absolute turn angle $|\tau_j|$ of vertex v_j has a limit $|\tau_j^*|$. If $|\tau_j^*| > 0$ (i.e., v_j^* is not a hairpin in all limit points P^*), then by Lemma 1, $|\tau_j^t| \geq |\tau_j^*| > 0$. For sufficiently large t, v_{j-1}^t approaches the line $\ell_{i,j}^t$. To form the absolute turn angle $|\tau_j^t| \geq |\tau_j^*| > 0$ at v_j, v_{j+1}^t must eventually be bounded away from the line $\ell_{i,j}^t$: after some time T, the minimum of the two angles between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ must be bounded below by some $\alpha > 0$. Now suppose that some $v_k^{t_r-1}$ were to extend beyond $v_j^{t_r-1}$ in the projection onto the line $\ell_{i,j}^{t_r-1}$ for some $t_r - 1 > T$. As illustrated in Figure 1, for the deflation of the chain $v_i^{t_r-1}, v_{i+1}^{t_r-1}, \ldots, v_j^{t_r-1}$ to not cause the next polygon P^{t_r} to self-intersect, the minimum of the two angles between $v_j^{t_r-1} v_k^{t_r-1}$ and $\ell_{i,j}^{t_r-1}$ must also be at least α.

But this is impossible for sufficiently large t, because v_k^t accumulates on the line $\ell_{i,j}^t$. Hence, in fact, v_k^t must not extend beyond v_j^t in the $\ell_{i,j}^t$ projection for sufficiently large t. In other words, when v_j^* is not a hairpin, each v_k^* must not extend beyond v_j^* on the line $\ell_{i,j}^*$. A symmetric argument handles the case when v_i^* is not a hairpin.

Finally, suppose that $j - i = 2$. In this case, because $v_{i+1}^* = v_{j-1}^*$ is a hairpin, it must extend beyond one of its neighbors, v_i^* or v_j^*. By the argument above, in the first case, v_i^* must be a hairpin, and in the second case, v_j^* must be a hairpin. Thus, as desired, either v_i^* or v_j^* must be a hairpin. $\square$

4 Deflating Quadrilaterals

We briefly review facts about quadrilateral deflation proved by Fevens et al. [6] and Ballinger [1]. For completeness, we also show how to prove these results using, in particular, our new Lemma 5.



(a) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is less than α, hence in the next deflation step the chain $v_i^t \ldots v_j^t$ will intersect the polygon.

(b) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is greater than α, so the polygon will not self-intersect in the next deflation step.

Fig. 1. Because v_j^t is not a hairpin, the minimum angle α between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ is strictly positive. If any vertex v_k^t of the chain $v_i^t, v_{i+1}^t, \ldots, v_j^t$ extends beyond v_j^t, then the minimum angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ must be at least α for the next deflation step P^{t+1} to not self-intersect. The dotted curve represents the rest of the polygon chain and the shaded area is the polygon interior below line $\ell_{i,j}^t$.

Lemma 6. [1] *Any accumulation point of an infinite deflation sequence of a quadrilateral is flat and has no straight vertices.*

Proof. First we argue that all quadrilaterals $P^1, P^2, \ldots$ (excluding the initial quadrilateral P^0) have no straight vertices. Because deflations are the inverse of pocket flips, and pocket flips do not exist for convex polygons, deflation always results in a nonconvex polygon. Thus all quadrilaterals P^t with $t > 0$ must be nonconvex. Hence no P^t with $t > 0$ can have a straight vertex, because then it would lie along an edge of the triangle of the other three vertices, making the quadrilateral convex. By Corollary 1, there are also no straight vertices in any accumulation point P^*.

By Lemma 4, there is a subchain $v_i, v_{i+1}, \ldots, v_j$, where $j - i \geq 2$, that is the pocket chain of infinitely many deflations. In fact, $j - i$ must equal 2, because reflecting a longer (4-vertex) pocket chain would not change the polygon. Applying Lemma 5 to $P^1, P^2, \ldots$ (where there are no straight vertices), for any accumulation point P^*, v_{i+1}^* is a hairpin and either v_i^* or $v_j^* = v_{i+2}^*$ is a hairpin. Hairpin v_{i+1}^* implies that v_i^*, v_{i+1}^*, v_{i+2}^* are collinear, while hairpin v_i^* or v_{i+2}^* implies that the remaining vertex $v_{i+3}^* = v_{i-1}^*$ lie on that same line. Therefore, any accumulation point P^* is flat. □

Theorem 1. [6, 1] *A simple quadrilateral with side lengths $\ell_1, \ell_2, \ell_3, \ell_4$ is infinitely deflatable if and only if*

1. *opposite edges sum equally, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$; and*
2. *adjacent edges differ, i.e., $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, $\ell_4 \neq \ell_1$.*

Furthermore, every such infinitely deflatable quadrilateral deflates infinitely independent of the choice of deflation sequence.

Proof. Fevens et al. [6] proved that every quadrilateral satisfying the two conditions on its edge lengths is infinitely deflatable, no matter which deflation sequence we make. Thus the two conditions are sufficient for infinite deflation.

To see that the first condition is necessary, we use Lemma 6. Because deflation preserves edge lengths, so do accumulation points of an infinite deflation sequence, so the flat limit configuration from Lemma 6 is a flat configuration of the edge lengths $\ell_1, \ell_2, \ell_3, \ell_4$. By a suitable rotation, we may arrange that the flat configuration lies along the x axis. By Lemma 6, no vertex is straight, so every vertex must be a hairpin. Thus, during a traversal of the polygon boundary, the edges alternate between going left ℓ_i and going right ℓ_i. At the end of the traversal, we must end up where we started. Therefore, $\pm(\ell_1 - \ell_2 + \ell_3 - \ell_4) = 0$, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$.

To see that the second condition is necessary, suppose for contradiction that $\ell_1 = \ell_2$ (the other contrary cases are symmetric). By the first condition, $\ell_1 + \ell_3 = \ell_2 + \ell_4$, so $\ell_3 = \ell_4$. Thus, the polygon is a kite, having two pairs of adjacent equal sides. (Also, all four sides might be equal.) Every kite has a chord that is a line of reflectional symmetry. No kite can deflate along this line, because such a deflation would cause edges to overlap with their reflections. If a kite is convex, it may deflate along its other chord, but then it becomes nonconvex, so it can be deflated only along its line of reflectional symmetry. Therefore, a kite can be deflated at most once, so any infinitely deflatable quadrilateral must have $\ell_1 \neq \ell_2$ and symmetrically $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, and $\ell_4 \neq \ell_1$. □

5 Deflating Pentagons

First we observe that the pentagon problem is relatively simple if we allow a straight vertex: we can subdivide the long edge of an infinitely deflating quadrilateral.

Theorem 2. *There is a simple pentagon with a straight vertex that deflates infinitely for all deflation sequences, exactly like the quadrilateral on the nonflat vertices.*

Proof. See Figure 2. We start with an infinitely deflating quadrilateral $\langle v_0, v_1, v_2, v_3 \rangle$ according to Theorem 1, and add a straight vertex v_4 along the edge v_3v_0. As long as we never deflate along a line passing through the straight vertex v_4, the deflations act exactly like the quadrilateral, and thus continue infinitely no matter which deflation sequence we choose. To achieve this property, we set the length of segment v_3v_0 to 1, with v_4 at the midpoint; we set the lengths of edges v_0v_1 and v_2v_3 to 2/3; and we set the length of edge v_1v_2 to 1/3. Then we deflate the quadrilateral until the vertices are so close to being hairpins that v_4 cannot see the nonadjacent convex vertex and the line through v_4 and the reflex vertex intersects the pentagon at another point. Thus no line of deflation passes through v_4, so we maintain infinite deflation as in the induced quadrilateral. □

Finally we show that any infinitely deflating pentagon induces an infinitely deflating quadrilateral.

Theorem 3. *Every simple pentagon with no straight vertices can be deflated by a finite sequence of (well-chosen) deflations. Furthermore, any infinite deflation sequence in such a pentagon induces an infinitely deflating quadrilateral.*

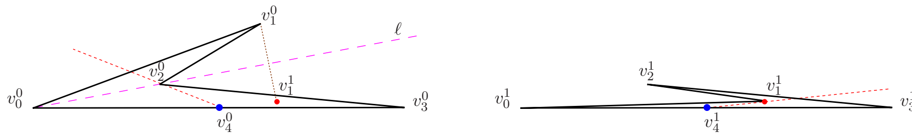


Fig. 2. An infinitely deflatable pentagon that induces an infinitely deflatable quadrilateral (left) and its configuration after the first deflation (right).

Proof. Let P be a pentagon with no straight vertices, and assume for the sake of contradiction that P deflates infinitely. Consider any accumulation point P^* of an infinite deflation sequence $P^0, P^1, P^2, \ldots$. By Lemma 4, there is an infinitely deflating pocket chain, say $v_0, v_1, \ldots, v_j$, where $j \geq 2$. By Lemma 5, $v_1^*, \ldots, v_{j-1}^*$ are hairpin vertices. Because the pentagon has only five vertices, $j \leq 4$. In fact, $j \leq 3$: if $j = 4$, this pocket chain would encompass all five vertices, making P^* collinear, which contradicts Lemma 3. If $j = 3$, then v_1^* and v_2^* are hairpins. If $j = 2$, then by Lemma 5, either v_0^* or v_2^* must be a hairpin; assume by symmetry that it is v_2^*. Thus, in this case, again v_1^* and v_2^* are hairpins. Hence, in all cases, v_1^* and v_2^* are hairpins, so $v_0^*, v_1^*, v_2^*, v_3^*$ are collinear, while by Lemma 3 the fifth vertex v_4^* must be off this line. In particular, v_0^*, v_3^*, and v_4^* are not hairpins.

By Lemma 5, any infinitely deflating chain is flat in the accumulation point P^*, so the only possible infinitely deflating chains are v_0, v_1, v_2; v_1, v_2, v_3; and v_0, v_1, v_2, v_3 (Figure 3). Let T denote the time after which only these three chains deflate. Thus, after time T, v_0, v_3, and v_4 stop moving, so in particular, v_4's angle and the length of the edge v_0v_3 take on their final values. Therefore, after time T, the vertices v_0, v_1, v_2, v_3 induce a quadrilateral that deflates infinitely, except that the chain v_0, v_1, v_2, v_3 might deflate. However, if at some time $t > T$ the chain $v_0^t, v_1^t, v_2^t, v_3^t$ deflates along the line through v_0^t and v_3^t into the triangle $v_0^t v_3^t v_4^t$, then the convex hull of P^{t+1} is $v_0^{t+1} v_3^{t+1} v_4^{t+1}$, which is fixed, so no further deflations are possible, resulting in a finite deflation sequence. Therefore the infinite deflation sequence can deflate only the chains v_0, v_1, v_2 and v_1, v_2, v_3 after time T. Indeed, after time T the sequence must alternate between deflating these two chains, because no chain can deflate twice in a row.

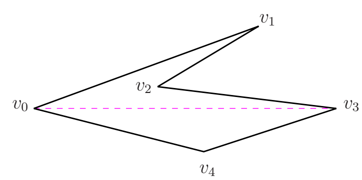
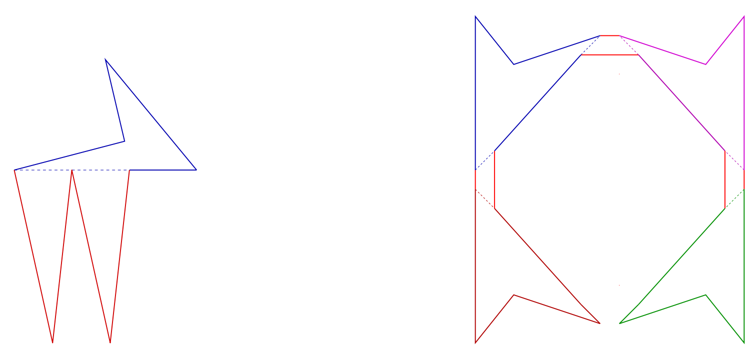


Fig. 3. A pentagon with an induced infinitely deflating quadrilateral, which is infinitely deflatable if we deflate only the subchain v_0, v_1, v_2, v_3.

We claim that v_1^* and v_2^* lie along the segment $v_0^* v_3^*$. Because v_1^* and v_2^* are hairpins, the only other possibilities are that v_1^* extends beyond v_3^* or that v_2^* extends beyond v_0^*. If v_1^* extended beyond v_3^*, then applying Lemma 5 to v_1, v_2, v_3 would imply that v_3^* is a hairpin, which is a contradiction. Therefore, v_1^* must lie along the segment $v_0^* v_3^*$, and similarly v_2^* must lie along the segment $v_0^* v_3^*$. By Theorem 1, no two adjacent edges of the quadrilateral have the same length, so in fact v_1^* and v_2^* must be strictly interior to the segment $v_0^*v_3^*$. Hence, for sufficiently large $t > T$, $v_0^t, v_1^t, v_2^t, v_3^t$ are arbitrarily close to collinear with v_1^t and v_2^t projecting to the relative interior of segment $v_0^tv_3^t$. Also, v_1^t and v_2^t must be outside the triangle $v_0^tv_3^tv_4^t$ because the quadrilateral v_0, v_1, v_2, v_3 remains deflatable. As a consequence, for sufficiently large $t > T$, we can deflate the chain $v_0^t, v_1^t, v_2^t, v_3^t$, which prevents all further deflations as argued above. Thus we obtain an alternate, finite deflation sequence. □

6 Larger Polygons and Well-Chosen Deflations

It is easy to construct n-gons with $n \geq 6$ that deflate infinitely, no matter which deflation sequence we choose. See Figure 4(a) for the idea of the construction. We can add any number of spikes to an infinitely deflating quadrilateral to obtain n-gons with $n \geq 6$ and even. For $n \geq 7$ and odd, we can shave off the tip of one of the spikes. Thus, $n = 5$ is the only value for which every n-gon with no straight vertices can be finitely deflated.



(a) An infinitely deflating octagon constructed by adding long spikes to an infinitely deflating quadrilateral.

(b) An infinitely deflating 18-gon constructed from four infinitely deflating quadrilaterals.

Fig. 4. Infinitely deflating polygons by combining infinitely deflating quadrilaterals.

None of the infinitely deflating polygons of Figure 4 are particularly satisfying because their accumulation points are not flat. Are there any n-gons, $n > 4$, that have no straight vertices and always deflate infinitely to flat accumulation points?

If we require that the n-gon is infinitely deflatable to a flat accumulation point only for at least one deflation sequence, then we can construct such a hexagon by taking two infinitely deflating quadrilaterals v_0, v_1, v_2, v_3 and v_3, v_4, v_5, v_0 (with their longest edge having the same length) and joining them along their longest edge; removing this edge will leave us with hexagon $v_0, v_1, v_2, v_3, v_4, v_5$. See Figure 5. This hexagon will deflate infinitely if we deflate only the two subchains v_0, v_1, v_2, v_3 and v_3, v_4, v_5, v_0 independently, and never deflate across the line through v_0 and v_3. This hexagon has an infinitely deflating quadrilateral

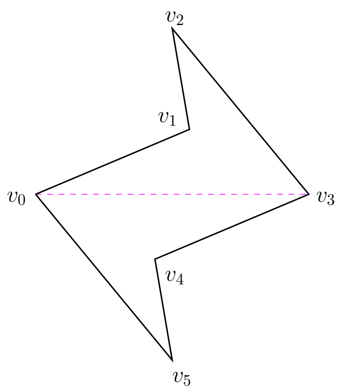


Fig. 5. An infinitely deflating hexagon constructed by joining two infinitely deflating quadrilaterals along their longest edge.

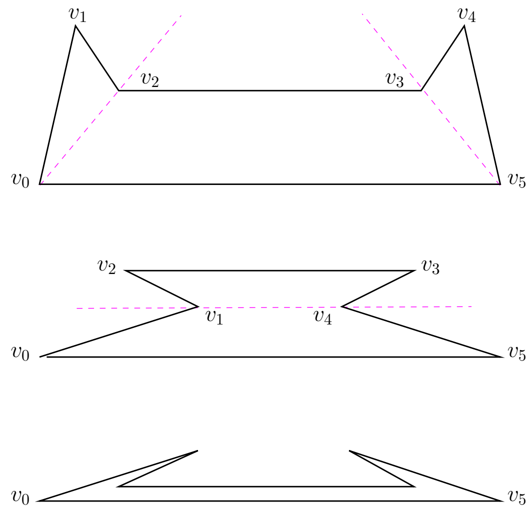


Fig. 6. A hexagon that deflates infinitely for a well-chosen deflation sequence but induces no infinitely deflating quadrilateral.

as a subpolygon, and indeed its infinite deflation sequences are interleavings of the two such quadrilaterals.

Next we present a family of hexagons that deflate infinitely to a flat accumulation point for some deflation sequence but do not induce an infinitely deflating quadrilateral. Figure 6 shows an example.

Theorem 4. *A simple hexagon $H = \langle v_0, v_1, v_2, v_3, v_4, v_5 \rangle$ with side lengths $\ell_i = |v_{i-1}v_i|$ (where $v_6 = v_0$) has an infinite deflation sequence with flat accumulation points if it satisfies the following five properties:*

1. *opposite edges sum equally, i.e., $\ell_1 + \ell_3 + \ell_5 = \ell_2 + \ell_4 + \ell_6$;*
2. *adjacent edges differ, i.e., $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, $\ell_4 \neq \ell_5$, $\ell_5 \neq \ell_6$, $\ell_6 \neq \ell_1$;*
3. *$\frac{1}{2}\ell_1 < \ell_2 < \ell_1$;*
4. *$\ell_6 > 3\ell_1$; and*
5. *the hexagon is symmetric about the perpendicular bisector of the edge v_0v_5. (In particular, $\ell_1 = \ell_5$ and $\ell_2 = \ell_4$, and v_0v_5 is parallel to v_2v_3.)*

Proof. Consider a hexagon H satisfying the five properties. Assume by suitable rotation and reflection that v_0v_5 (and hence v_2v_3) is horizontal, v_0 is left of v_5, and v_2 (and hence v_3) is above the horizontal line through v_0 and v_5.

We argue that any such hexagon H is simple. Obviously, the parallel edges v_2v_3 and v_0v_5 do not cross. If v_0v_1 (and hence v_4v_5) intersects v_2v_3, as in Figure 7(a), then by the planar quadrilateral uncrossing lemma, $\ell_1+\ell_3 > \ell_2+|v_0v_3|$

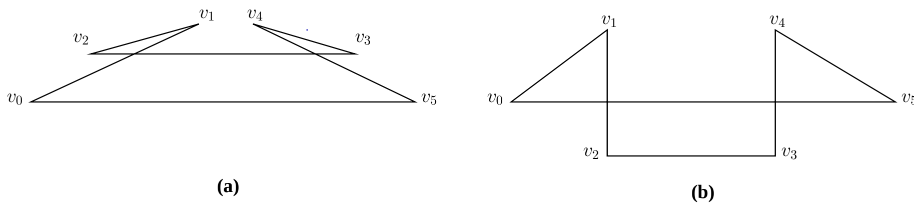


Fig. 7. The two possible configurations of H if it self-intersects.

and $\ell_5+|v_0v_3| > \ell_4+\ell_6$, which sum to $\ell_1+\ell_3+\ell_5+|v_0v_3| > \ell_2+\ell_4+\ell_6+|v_0v_3|$, contradicting Property 1. Similarly, if v_1v_2 (and hence v_3v_4) intersects v_0v_5, as in Figure 7(b) or its reflection, then $\ell_2+\ell_6 > \ell_1+|v_2v_5|$ and $\ell_4+|v_2v_5| > \ell_3+\ell_5$, which sum to $\ell_2+\ell_4+\ell_6+|v_2v_5| > \ell_1+\ell_3+\ell_5+|v_2v_5|$, contradicting Property 1. In projection onto the horizontal line through v_0v_5, v_1 can reach at most ℓ_1 to the right of v_0 and v_4 can reach at most $\ell_5 = \ell_1$ to the left of v_5. By Property 4, this travel is small enough that v_1 must be left of v_4. Thus, in particular, v_0v_1 cannot cross v_4v_5. If v_2 were right of v_3, then $|v_0v_2|+|v_3v_5| > \ell_3+\ell_6$, so by the triangle inequality, $\ell_1+\ell_2+\ell_4+\ell_5 > \ell_3+\ell_6$, so by Property 4, $\ell_1+\ell_2+\ell_4+\ell_5 > \ell_3+3\ell_1$, i.e., $\ell_2+\ell_4 > \ell_3+\ell_1$, so by Property 1, $\ell_6 < \ell_5$, contradicting Property 4. Hence, v_2 is left of v_3. Thus v_0, v_1, and v_2 are left of v_3, v_4, and v_5, so v_0v_1 and v_1v_2 cannot cross v_3v_4 or v_4v_5. Hence no pairs of edges can cross. Property 2, together with Properties 1 and 5, forbids edges from overlapping and forbids nonadjacent edges from touching. Therefore H must be simple.

Next we claim that v_1 (and hence v_4), like v_2 and v_3, is above the horizontal line through v_0v_5, implying that v_0 (and hence v_5) is convex. Because $\ell_1 = \ell_5$ and $\ell_2 = \ell_4$, Property 1 can be rewritten as $2\ell_1 + \ell_3 = 2\ell_2 + \ell_6$. By Property 3, $\ell_2 < \ell_1$, so $\ell_3 < \ell_6$. Thus v_2 is above and to the right of v_0. Because $\ell_2 < \ell_1$, if v_1 were not also above v_0, the edge v_1v_2 could not reach a point above and to the right of v_0 without crossing v_0v_5. But we showed that H is simple, so v_1 must in fact be above v_0.

Now we claim that the hexagon H deflates infinitely by repeating the following three-step sequence ad infinitum: first deflate across the line passing through v_0 and v_2, second across the line through v_3 and v_5, and third across the line through v_2 and v_3. Exactly where we begin this infinite sequence depends on the initial hexagon H: if v_2 (and hence v_3) is reflex, we start on the first step; otherwise, we start on the third step. In general, the first step will be executed when v_2 (and v_3) is reflex, the second step will be executed when just v_3 is reflex, and the third step will be executed when v_2 and v_3 are convex. We also maintain the invariant that the hexagon is symmetric about the perpendicular bisector of v_0v_5 (Property 5) after every execution of the second and third steps. We need to show that (1) no deflation step introduces crossings, and (2) every line of deflation intersects the hexagon only at the two vertices defining it.

We have already shown that the hexagon is simple after any execution of the second or third step, because then the hexagon satisfies Property 5. We can argue simplicity after the execution of the first step by comparing with the hexagon that was just before the first step and with the hexagon that will be just after the next second step. Therefore the hexagon is simple at all stages.

It remains to show that every line of deflation hits the hexagon boundary just at its two defining vertices. The argument for the first step, deflating across v_0v_2, is below. The argument for the second step is similar to simplicity: Properties 3 and 4 guarantee that v_1 is always left of v_3, and in this case v_1 is below the horizontal line through v_3, while v_0 is below and right of v_3, so the line through v_3 and v_5 cannot hit v_0v_1 or v_1v_2. The argument for the third step is easy: the line through v_1 and v_4 cannot hit any of the incident edges (v_0v_1, v_1v_2, v_3v_4, and v_4v_5), and by Property 5 the line is horizontal, so it cannot hit the two remaining horizontal edges (v_2v_3 and v_0v_5).

Finally we consider deflating across v_0v_2, where it suffices to prove that v_4 is to the right of the line from v_0 to v_2. Assume by suitable translation that vertex v_0 is at the origin, and let θ be the interior angle at v_0. Then v_1 has coordinates $\langle \ell_1 \cos\theta, \ell_1 \sin\theta \rangle$ and $v_4 = \langle \ell_6 - \ell_1 \cos\theta, \ell_1 \sin\theta \rangle$. The x coordinate of v_2 is $\frac{1}{2}\ell_6 - \frac{1}{2}\ell_3$, which by adding half of Property 1 is $\ell_1 - \ell_2$. Now consider the right triangle v_1v_2x, where x is the point below v_1 and horizontal with v_2. The hypotenuse is ℓ_2, and the horizontal edge has length $(\ell_1 - \ell_2) - \ell_1 \cos\theta = \ell_1(1-\cos\theta) - \ell_2$, so the vertical edge has length $\sqrt{\ell_2^2 - (\ell_1(1-\cos\theta) - \ell_2)^2}$. Thus, $v_2 = \langle \ell_1 - \ell_2, \ell_1 \sin\theta - \sqrt{\ell_2^2 - (\ell_1(1-\cos\theta) - \ell_2)^2} \rangle$. Note that, for v_2 to have a valid (noncomplex) solution, we must have $2\ell_2 > \ell_1$, which is part of Property 3.

Now, v_4 is to the right of the line from v_0 to v_2 if and only if the signed area of the triangle $v_0v_2v_4$ is negative. Thus we desire the following inequality:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \ell_1 - \ell_2 & \ell_1 \sin\theta - \sqrt{\ell_2^2 - (\ell_1(1-\cos\theta) - \ell_2)^2} & 1 \\ \ell_6 - \ell_1 \cos\theta & \ell_1 \sin\theta & 1 \end{vmatrix} < 0.$$

After significant simplification, this inequality becomes

$$\ell_1(\cos\theta - 1)(\ell_1 - \ell_2)[\ell_1^2(1 + 3\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 - \ell_1\ell_2(1 + \cos\theta)] < 0.$$

Because θ is between 0 and π, and $\ell_2 < \ell_1$, this inequality is equivalent to

$$\ell_1^2(1 + 3\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 - \ell_1\ell_2(1 + \cos\theta) > 0.$$

Also, because $\ell_2 < \ell_1$, it is enough to show

$$\ell_1^2(2\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 > 0.$$

If $\ell_6 = \alpha\ell_1$, then the inequality becomes

$$(\cos\theta + 2\cos^2\theta) - \alpha(1 + 3\cos\theta) + \alpha^2 > 0.$$

The maximum lower bound on α that satisfies this inequality occurs at $\theta = 0$; in this case, we obtain $3 - 4\alpha + \alpha^2 = 0$, which has solution $\alpha = 3$. Therefore, Condition 4 that $\ell_6 > 3\ell_1$ suffices.

We can easily show that every accumulation point of our deflation sequence is flat: because each of the chains v_0, v_1, v_2; v_3, v_4, v_5; and v_1, v_2, v_3, v_4 deflate infinitely, then by Lemma 5, in every accumulation point, the vertices of each of the chains are collinear, forcing all six vertices to be collinear. □

7 Open Problems

It remains open whether there exist n-gons, $n \geq 6$, that have no straight vertices and deflate infinitely for every deflation sequence to flat accumulation points. Also, does every infinite deflation sequence have a (unique) limit? Our proofs would likely simplify if we knew there were only one accumulation point.

Is there an efficient algorithm to determine whether a given polygon P has an infinite deflation sequence? What about detecting whether all deflation sequences are infinite? Even given a (succinctly encoded) infinite sequence of deflations, can we efficiently determine whether the sequence is valid?

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