**Algorithmica manuscript No.**
(will be inserted by the editor)

# Diameter and Treewidth in Minor-Closed Graph Families, Revisited

**Erik D. Demaine, MohammadTaghi Hajiaghayi**

MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar St., Cambridge, MA 02139, USA. {edemaine,hajiagha}@mit.edu

The date of receipt and acceptance will be inserted by the editor

**Abstract** Eppstein [5] characterized the minor-closed graph families for which the treewidth is bounded by a function of the diameter, which includes, e.g., planar graphs. This characterization has been used as the basis for several (approximation) algorithms on such graphs (e.g., [2,5–8]). The proof of Eppstein is complicated. In this short paper we obtain the same characterization with a simple proof. In addition, the relation between treewidth and diameter is slightly better and explicit.

**Key words** apex graphs, graph minors, bounded local treewidth, graph algorithms, approximation algorithms

## 1 Introduction

Eppstein [5] introduced the *diameter-treewidth property* for a class of graphs, which requires that the treewidth of a graph in the class is upper bounded by a function of its diameter. This notion has been used extensively in a slightly modified form called the *bounded-local-treewidth property*, which requires that the treewidth of any connected subgraph of a graph in the class is upper bounded by a function of its diameter. For minor-closed graph families, which is the focus of most work in this context, these properties are identical.

The reason for introducing graphs of bounded local treewidth is that they have many similar properties to both planar graphs and graphs of bounded treewidth, two classes of graphs on which many problems are substantially easier. In particular, Baker's approach for polynomial-time approximation schemes (PTASs) on planar graphs [1] applies to this setting. As a result, PTASs are known for hereditary maximization problems such as maximum independent set, maximum triangle matching, maximum $H$-matching, maximum tile salvage, minimum vertex cover, minimum dominating set, minimum edge-dominating set, and subgraph isomorphism for a fixed pattern [2,5,8]. Graphs of bounded local treewidth also admit several efficient fixed-parameter algorithms. In particular, Frick and Grohe [6] give a general framework for deciding any property expressible in first-order logic in graphs of bounded local treewidth.

The foundation of these results is the following characterization by Eppstein [5] of minor-closed families with the diameter-treewidth property. An *apex graph* is a graph in which the removal of some vertex leaves a planar graph.

**Theorem 1** *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ has the diameter-treewidth property if and only if $\mathcal{F}$ does not contain all apex graphs, i.e., $\mathcal{F}$ excludes some apex graph.*

We reprove this theorem with a much simpler proof. Similar to Eppstein's proof, we use the following theorems from Graph Minor Theory. The *$m \times m$ grid* is the planar graph with $m^2$ vertices arranged on a square grid and with edges connecting horizontally and vertically adjacent vertices.

**Theorem 2** [4] *For integers $r$ and $m$, let $G$ be a graph of treewidth at least $m^{4r^2(m+2)}$. Then $G$ contains either the complete graph $K_r$ or the $m \times m$ grid as a minor.*

**Theorem 3** [9] *Every planar graph $H$ can be obtained as a minor of the $r \times r$ grid $H$, where $r = 14|V(H)| - 24$.*

For readers unfamiliar with the notions of minors and treewidth, we give the necessary definitions for background.

*Contracting* an edge $e = \{u, v\}$ in an (undirected) graph $G$ is the operation of replacing both $u$ and $v$ by a single vertex $w$ whose neighbors are all vertices that were neighbors of $u$ or $v$, except $u$ and $v$ themselves. A graph $G$ is a *minor* of a graph $H$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. A graph class $\mathcal{C}$ is *minor-closed* if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *$H$-minor-free* if $H \notin \mathcal{C}$.

Representation of a graph as a tree with a *tree decomposition* plays an important role in the design of algorithms. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\chi, T)$ where $T = (I, F)$ is a tree and $\chi = \{\chi_i \mid i \in I\}$ is a family of subsets of $V(G)$ such that (1) $\bigcup_{i \in I} \chi_i = V$; (2) for each edge $e = \{u, v\} \in E$, there exists an $i \in I$ such that both $u$ and $v$ belong to $\chi_i$; and (3) for all $v \in V$, the set of nodes $\{i \in I \mid v \in \chi_i\}$ forms a connected subtree of $T$. The maximum size of any $\chi_i$, minus one, is called the *width* of the tree decomposition. The *treewidth* of a graph $G$, denoted by $\mathbf{tw}(G)$, is the minimum width over all possible tree decompositions of $G$.

## 2 The Main Result

First we present a property of apex-minor-free graphs. The vertices $(i, j)$ of the $m \times m$ grid with $i \in \{1, m\}$ or $j \in \{1, m\}$ are called *boundary* vertices, and the rest of the vertices in the grid are called *nonboundary* vertices.

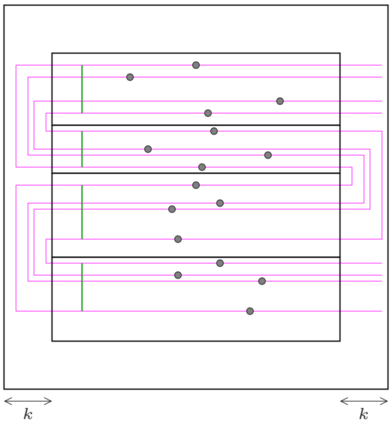


**Fig. 1** Construction of the minor $k \times k$ grid $K$.

**Lemma 1** *Let $G$ be an $H$-minor-free graph for an apex graph $H$, let $k = 14|V(H)|-22$, and let $m > 2k$ be the largest integer such that $\mathbf{tw}(G) \geq m^{4|V(H)|^2(m+2)}$. Then $G$ can be contracted into an* augmented grid *$R$, i.e., an $(m-2k)\times(m-2k)$ grid augmented with additional edges (and no additional vertices) such that each vertex $v \in V(R)$ is adjacent to less than $(k+1)^6$ nonboundary vertices of the grid.*

*Proof* By Theorem 2, $G$ contains an $m \times m$ grid $M$ as a minor. Thus there exists a sequence of edge contractions and edge/vertex deletions reducing $G$ to $M$. We apply to $G$ the edge contractions from this sequence; we ignore the edge deletions; and instead of deletion of a vertex $v$, we only contract $v$ into one of its neighbors. Call the new graph $G'$, which has the $m \times m$ grid $M$ as a subgraph and in addition $V(G') = V(M)$.

We claim that each vertex $v \in V(G')$ is adjacent to at most $k^4$ vertices in the central $(m-2k)\times(m-2k)$ subgrid $M'$ of $M$. In other words, let $N$ be the set of neighbors of any vertex $v \in V(G')$ that are in $M'$. We claim that $|N| \leq k^4$. Suppose for contradiction that $|N| > k^4$.

Let $n_x$ denote the number of distinct $x$ coordinates of the vertices in $N$, and let $n_y$ denote the number of distinct $y$ coordinates of the vertices in $N$. Thus, $|N| \leq n_x \cdot n_y$. Assume by symmetry that $n_y \geq n_x$, and therefore $n_y \geq \sqrt{|N|} > k^2$.

We define the subset $N'$ of $N$ by removing all but one (arbitrarily chosen) vertex that share a common $y$ coordinate, for each $y$ coordinate. Thus, all $y$ coordinates of the vertices in $N'$ are distinct, and $|N'| = n_y > k^2$. We discard all but $k^2$ (arbitrarily chosen) vertices in $N'$ to form a slightly smaller set $N''$. We divide these $k^2$ vertices into $k$ groups each of exactly $k$ consecutive vertices according to the order of their $y$ coordinates. Now we construct the minor $k\times k$ grid $K$ as shown in Figure 1. Because each $y$ coordinate is unique, we can draw long horizontal segments through every point. The $k$ columns on the left-hand and right-hand sides of $M$ allow us to connect these horizontal segments together into $k$ vertex-disjoint paths, each passing through exactly $k$ vertices of $N''$. These paths can be con-

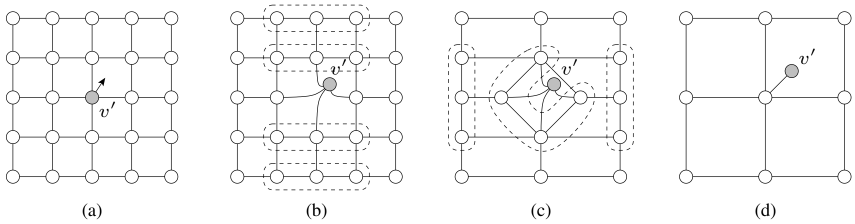


**Fig. 2** In the $k \times k$ grid $K$, we (a) lift the vertex $v'$, (b) contract the adjacent columns, and (c) contract the adjacent rows, to form a $(k-2) \times (k-2)$ grid $K'$. Vertex $v'$ is adjacent to all vertices in the grid, though the figure just shows four neighbors for visibility.

nected by vertical segments within each group. This arrangement of paths has the desired $k \times k$ grid $K$ as a minor, where the vertices of the grid correspond to the vertices in $N''$.

Now, if $v$ has been used in the contraction of a vertex $v'$ in $K$, we proceed as shown in Figure 2. First we "lift" $v'$ from the grid—not removing it from the graph per se, but marking it as "outside the grid." Then we contract the remainder of $v'$'s column and the two adjacent columns (if they exist) into a single column. Similarly we contract the remainder of $v'$'s row with the adjacent rows. Thus we obtain as a minor of $K$ a $(k-2) \times (k-2)$ grid $K'$ such that vertex $v'$ is outside this grid and adjacent to all vertices of the grid. Now, by Theorem 3, because $k-2 \geq 14|V(H)|-24$, we can consider $v'$ as the apex of $H$ and obtain the planar part of $H$ as a minor of $K'$. Hence the original graph $G$ is not $H$-minor-free, a contradiction. This concludes the proof of the claim that $|N| \leq k^4$.

Finally, form a new graph $R$ by taking graph $G'$ and contracting all $2k$ boundary rows and $2k$ boundary columns into two boundary rows and two boundary columns (one on each side). The number of neighbors of each vertex of $R$ that are not on the boundary is at most $(k+1)^2k^4$. The factor $(k+1)^2$ is for the boundary vertices each of which is obtained by contraction of at most $(k+1)^2$ vertices. $\square$

Now we are ready to prove Theorem 1.

*Proof (of Theorem 1)* One direction is easy. The apex graphs $A_i$, $i = 1, 2, \ldots$, obtained from the $i \times i$ grid by connecting a new vertex $v$ to all vertices of the grid have diameter two and treewidth $i+1$, because the treewidth of the $i \times i$ grid is $i$ (see, e.g., [3]). Thus a minor-closed family of graphs with the diameter-treewidth property cannot contain all apex graphs. Next consider the other direction. Let $G$ be a graph from a minor-closed family $\mathcal{F}$ of graphs excluding an apex graph $H$. We show that the treewidth of $G$ is bounded above by a function of $|V(H)|$ and its diameter $d$. Let $m$ be the largest integer such that $\mathbf{tw}(G) \geq m^{4|V(H)|^2(m+2)}$, and let $k = 14|V(H)| - 22$. Let $R$ be the $(m-2k) \times (m-2k)$ augmented grid obtained from $G$ by contraction, using Lemma 1. Because diameter does not increase by contraction, the diameter of $R$ is at most $d$. In addition, one can easily observe that the number of vertices of distance at most $i$ from any vertex in $R$ is at most $4r + 4r(k+1)^6 + 4r(k+1)^{12} + \cdots + 4r(k+1)^{6i} \leq 4r(k+1)^{6(i+1)}$, where $r = m-2k$. Because the diameter is at most $d$, we have $4r(k+1)^{6(d+1)} \geq r^2$, i.e., $m \leq 2k + 4(k+1)^{6(d+1)}$. Thus the treewidth of $G$ is at most $(4(k+1)^{6(d+1)} + 2k + 1)^{4|V(H)|^2(4(k+1)^{6(d+1)}+2k+3)} = O(|V(H)|^{6(d+1)})^{O(|V(H)|^{6d+8})} = 2^{O(d|V(H)|^{6d+8} \lg |V(H)|)}$, as desired. $\square$

## References


1. B. S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. Assoc. Comput. Mach.*, 41(1):153–180, 1994.
2. E. D. Demaine, M. Hajiaghayi, N. Nishimura, P. Ragde, and D. M. Thilikos. Approximation algorithms for classes of graphs excluding single-crossing graphs as minors. *J. Comput. System Sci.* To appear.
3. R. Diestel. *Graph theory, Graduate Texts in Mathematics Vol. 173*. Springer-Verlag, New York, 2000.
4. R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen. Highly connected sets and the excluded grid theorem. *J. Combin. Theory Ser. B*, 75(1):61–73, 1999.
5. D. Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000.
6. M. Frick and M. Grohe. Deciding first-order properties of locally tree-decomposable graphs. *J. ACM*, 48(6):1184–1206, 2001.
7. M. Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.
8. M. Hajiaghayi and N. Nishimura. Subgraph isomorphism, log-bounded fragmentation and graphs of (locally) bounded treewidth. In *Proceedings of the 27th International Symposium on Mathematical Foundations of Computer Science (Poland, 2002)*, pages 305–318. 2002.
9. N. Robertson, P. D. Seymour, and R. Thomas. Quickly excluding a planar graph. *J. Combin. Theory Ser. B*, 62(2):323–348, 1994.