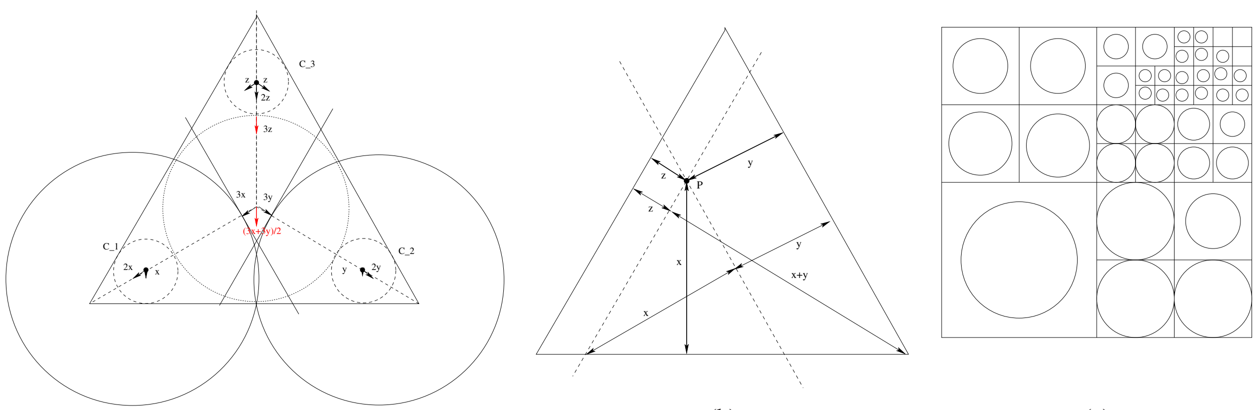
On the Complexity of Origami Design

Erik D. Demaine[*] Sándor P. Fekete[†] Robert J. Lang[‡]

The tree method of origami design [Lan96, LD06] has been a critical element in the growth of complex origami design over the past few decades. This design technique finds an optimal (largest scale) folding of a desired stick figure (metric tree) from a given rectangle of paper. The algorithm is efficient (polynomial time) aside from the first step, which places disks and rivers to reserve paper for flaps. In the special case of a star tree with varying edge lengths (i.e., designing an origami base with several adjacent flaps of different lengths), this problem is equivalent to packing n given disks in a given rectangle.

We prove for the first time that this disk-packing problem is NP-hard, showing that optimal origami design has to remain an art. Our proof is based on a reduction from 3-PARTITION, which requires partitioning a set of $3k$ numbers into k triples of sum at most 1. (A key step is hinted at in Figure 1 (a), while Figure 1 (b) shows a crucial lemma for the correctness of the reduction.)

On the positive side, we show that the design becomes a lot easier if one is willing to compromise on the size of the piece of paper. We prove that any given set of disks that fits into a unit square can easily and recursively be packed into a square of edge length $\frac{8}{\pi} = 2.546...$ (See Figure 1 (c) for the basic idea, which allows highly symmetric packings.) This is closely related to the minimum possible density of disks in a smallest enclosing square. A worst-case example shows this density can be as bad as 0.539..., while our bound establishes a lower bound of 0.154... Closing this gap is an interesting open problem.



(a) (b) (c)

Figure 1: (a) Reduction from 3-PARTITION: packing disks into triangular pockets. (b) A key lemma: in an equilateral triangle, the sum of altitudes is constant. (c) Approximating the required square size.

References

[Lan96] Robert J. Lang. A computational algorithm for origami design. In *Proceedings of the 12th Annual ACM Symposium on Computational Geometry*, pages 98–105, Philadelphia, PA, May 1996.

[LD06] Robert J. Lang and Erik D. Demaine. Facet ordering and crease assignment in uniaxial bases. In *Origami*^{4}*: Proceedings of the 4th International Conference on Origami in Science, Mathematics, and Education*, pages 189–205, Pasadena, California, September 2006. A K Peters.

[*] Computer Science and Artificial Intelligence Laboratory, MIT, USA, edemaine@mit.edu

[†] Dept. of Computer Science, Braunschweig Institute of Technology, Germany, s.fekete@tu-bs.de

[‡] Langorigami.com, robert@langorigami.com