k-Piece Dissection is NP-hard

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1 Introduction

We have known for centuries how to dissect any polygon P into any other polygon Q of equal area, that is, how to cut P into finitely many pieces and re-arrange the pieces to form Q [Fre97, Low14, Wal31, Bol33, Ger33]. But we know relatively little about how many pieces are necessary. For example, it is unknown whether dissecting a square into an equilateral triangle can be done with fewer than four pieces [Fre97]. Only recently was it established that a pseudopolynomial number of pieces suffices [ADD+11].

In this paper, we prove that minimizing the number of pieces is NP-hard. While unsurprising, this result is the first analysis of the complexity of dissection. We prove NP-hardness even when the polygons are restricted to be simple (hole-free) and orthogonal. The reduction holds independent of whether the dissections are restricted or not to making orthogonal cuts, and allowing or forbidding the pieces to rotate.

Our proof significantly strengthens the observation (originally made by the Demaines during JCDCG'98) that the second half of dissection—re-arranging given pieces into a target shape—is NP-hard: the special case of exact packing rectangles into rectangles can directly simulate 3-partition [DD07]. Effectively, the challenge in our proof is to construct a polygon for which any k-piece dissection must cut the polygon at locations we desire, so that we are left with a rectangle packing/3-partition problem.

2 Proof

Given a source polygon and a target polygon of equal area, the k-piece dissection problem is to decide if the source polygon can be cut into k connected pieces and packed into the target polygon. We prove k-piece dissection NP-hard even when the source and target polygons are simple orthogonal polygons.

Reduction. We reduce from 3-partition. Let A denote the set of n integers that constitutes the 3-partition instance and $p = \frac{3}{n}\Sigma A$ denote the partition sum. Let $d_s = \frac{4}{3}p + 1$ and $d_t = (n-1)d_s + \Sigma A + 2\max A$. We create a source polygon consisting of *element rectangles* of width a_i and height 1 for each $a_i \in A$ spaced d_s apart, connected below by a rectangular *bar* of width $\Sigma A + (\frac{n}{3} - 1)d_t$ and height $\epsilon < \frac{1}{k}$. The first element rectangle's left edge is flush with the left edge of the bar; the bar extends beyond the last element rectangle. Our target polygon consists of $\frac{n}{3}$ *partition rectangles* of width p and height 1 spaced d_t apart, connected by a bar of the same dimensions as the source polygon's bar. The first partition rectangle's left edge and last partition rectangle's right edge are flush with the ends of the bar. Both polygons' bars have the same area and the total area of the element rectangles equals the total area of the partition rectangles, so the polygons have the same area. The number of pieces $k = n$.

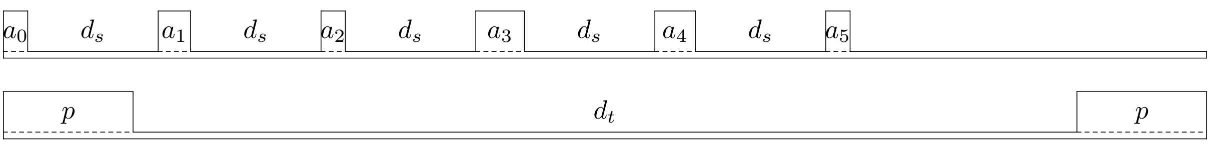

Figure 1: The source (above) and target (below) polygons for the 3-partition problem 30, 40, 30, 60, 50, 30, to scale.

3-partition $\implies$ Dissection. Given a 3-partition solution, we can cut all but the first element rectangle off the bar and pack them in the partition rectangles according to the 3-partition solution. This always puts the first element rectangle in the first partition rectangle, but the 3-partition solution is a set so we need not preserve the order of the partitions.

Dissection $\implies$ 3-partition. We can constrain the possible correct k-piece dissections to be of the form created by the transformation above, allowing us to recover a 3-partition solution from any valid dissection.

No piece in the dissection can contain portions of two element rectangles. Even if the element rectangles are adjacent, such a piece will not fit inside a partition rectangle because $d_s > (1+\epsilon)\sqrt{p^2+1}$ (the partition rectangle's diagonal, including protrusion into the bar). Even if the element rectangles are the first and last (most distant), such a piece cannot span two partition rectangles because $d_t > (n-1)d_s + 2\max A$. Every element rectangle must be contained in at least one piece, so each element rectangle is in exactly one piece. There are exactly as many pieces as element rectangles, so each piece contains all of exactly one element rectangle.

The bar must be in the same piece as the first element rectangle. Every piece containing part of the bar has height greater than 1 because it contains an entire element rectangle; thus the piece must protrude into the ϵ-tall portion of the target polygon, locking it in place with respect to the bar. Because $\epsilon < \frac{1}{k}$ and all pieces contain a whole element rectangle, pieces containing portions of the bar cannot be used to cover a partition rectangle as they cannot sum to an integral area. If there are multiple pieces containing part of the bar, they can exchange places, but no such piece can be placed further right than the last element rectangle. The last element rectangle ends at $(n-1)d_s + \Sigma A$, which is before the second partition rectangle begins at $p + d_t = p + (n-1)d_s + \Sigma A + 2\max A$, so the piece will not fit in the second partition rectangle. Because $d_s > p$, any piece beyond the first element rectangle will not fit in the first partition rectangle. Thus the bar must be entirely contained in the piece containing the first element rectangle.

These constraints force any k-piece dissection solution to cut all but the first element rectangle off the bar and pack them into the partition rectangles according to the 3-partition instance they represent. Thus we can read off the 3-partition solution from a k-piece dissection solution.

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