# Dissection with the Fewest Pieces is Hard, Even to Approximate

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**Abstract.** We prove that it is NP-hard to dissect one simple orthogonal polygon into another using a given number of pieces, as is approximating the fewest pieces to within a factor of $1 + 1/1080 - \varepsilon$.

## 1 Introduction

We have known for centuries how to dissect any polygon $P$ into any other polygon $Q$ of equal area, that is, how to cut $P$ into finitely many pieces and re-arrange the pieces to form $Q$ [7,13,14,2,11]. But we know relatively little about how many pieces are necessary. For example, it is unknown whether a square can be dissected into an equilateral triangle using fewer than four pieces [6,8, pp. 8–10]. Only recently was it established that a pseudopolynomial number of pieces suffices [1].

In this paper, we prove that it is NP-hard even to approximate the minimum number of pieces required for a dissection, to within some constant ratio. While perhaps unsurprising, this result is the first analysis of the complexity of dissection. We prove NP-hardness even when the polygons are restricted to be simple (hole-free) and orthogonal. The reduction holds for all cuts that leave the resulting pieces connected, even when rotation and reflection are permitted or forbidden.

Our proof significantly strengthens the observation (originally made by the Demaines during JCDCG'98) that the second half of dissection—re-arranging given pieces into a target shape—is NP-hard: the special case of exact packing rectangles into rectangles can directly simulate 3-Partition [5]. Effectively, the challenge in our proof is to construct a polygon for which any $k$-piece dissection must cut the polygon at locations we desire, so that we are left with a rectangle packing problem.

Due to the lack of space, we omit the proofs of some lemmas from this current version of our paper. For missing proofs, see the full version of this paper [3].

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## 2 The Problems

### 2.1 Dissection

We begin by formally defining the problems involved in our proofs, starting with $k$-PIECE DISSECTION, which is the central focus of our paper.

**Definition 1.** $k$-PIECE DISSECTION *is the following decision problem:*
INPUT: *two polygons $P$ and $Q$ of equal area, and a positive integer $k$.*
OUTPUT: *whether $P$ can be cut into $k$ pieces such that these $k$ pieces can be packed into $Q$ (via translation, optional rotation, and optional reflection).*

To prevent ill-behaved cuts, we require every piece to be a *Jordan region (with holes)*: the set of points interior to a Jordan curve $e$ and exterior to $k \geq 0$ Jordan curves $h_1, h_2, \ldots, h_k$, such that $e, h_1, h_2, \ldots, h_k$ do not meet. There are two properties of Jordan regions that we use in our proofs. First, Jordan regions are Lebesgue measurable; we will refer to the Lebesgue measure of each piece as its area. Second, a Jordan region is path-connected. For brevity, we refer to path-connected as connected throughout the paper.

Next we define the optimization version of the problem, MIN PIECE DISSECTION, in which the objective is to minimize the number of pieces.

**Definition 2.** MIN PIECE DISSECTION *is the following optimization problem:*
INPUT: *two polygons $P$ and $Q$ of equal area.*
OUTPUT: *the smallest positive integer $k$ such that $P$ can be cut into $k$ pieces such that these $k$ pieces can be packed into $Q$.*

### 2.2 5-Partition

Our NP-hardness reduction for $k$-PIECE DISSECTION is from 5-PARTITION, a close relative of 3-PARTITION.

**Definition 3.** 5-PARTITION *is the following decision problem:*
INPUT: *a multiset $A = \{a_1, \ldots, a_n\}$ of $n = 5m$ integers.*
OUTPUT: *whether $A$ can be partitioned into $A_1, \ldots, A_m$ such that, for each $i = 1, \ldots, m$, $\sum_{a \in A_i} a = p$ where $p = \left(\sum_{a \in A} a\right)/m$.*

Throughout the paper, we assume that the partition sum $p$ is an integer; otherwise, the instance is obviously a NO instance.

Garey and Johnson [9] originally proved NP-completeness of 3-PARTITION, a problem similar to 5-PARTITION except that 5 is replaced by 3. In their book [10], they show that 4-PARTITION is NP-hard; this result was, in fact, an intermediate step toward showing that 3-PARTITION is NP-hard. It is easy to reduce 4-PARTITION to 5-PARTITION and thus show it also NP-hard.[3]

Our reduction would work from 3-Partition just as well as 5-Partition. The advantage of the latter is that we can analyze the following optimization version.

[3] Given a 4-PARTITION instance $A = \{a_1, \ldots, a_n\}$, we can create a 5-PARTITION instance by setting $A' = \{na_1, \ldots, na_n, 1, \ldots, 1\}$ where the number of 1s is $n/4$.

**Definition 4.** MAX 5-PARTITION *is the following optimization problem:*

INPUT: *a multiset* $A = \{a_1, \ldots, a_n\}$ *of* $n = 5m$ *integers.*

OUTPUT: *the maximum integer* $m'$ *such that there exist disjoint subsets* $A_1, \ldots, A_{m'}$ *of* $A$ *such that, for each* $i = 1, \ldots, m'$, $\sum_{a \in A_i} a = p$ *where* $p = \frac{5}{n}\left(\sum_{a \in A} a\right)$.

### 2.3 Gap Problems

We show that our reductions have a property stronger than approximation preservation called *gap preservation.* Let us define the gap problem for an optimization problem, a notion widely used in hardness of approximation.

**Definition 5.** *For an optimization problem* $P$ *and parameters* $\beta > \gamma$ *(which may be functions of* $n$*), the* $\text{GAP}_P[\beta, \gamma]$ *problem is to distinguish whether the optimum of a given instance of* $P$ *is at least* $\beta$ *or at most* $\gamma$*. The input instance is guaranteed to not have an optimum between* $\beta$ *and* $\gamma$*.*

If $\text{GAP}_P[\beta, \gamma]$ is NP-hard, then it immediately follows that approximating $P$ to within a factor of $\beta/\gamma$ of the optimum is also NP-hard. This result makes gap problems useful for proving hardness of approximation.

## 3 Main Results

Now that we have defined the problems, we state our main results.

**Theorem 1.** $k$-PIECE DISSECTION *is NP-hard.*

We do not know whether $k$-PIECE DISSECTION is in NP (and thus is NP-complete). We discuss the difficulty of showing containment in NP in Section 7.

We also prove that the optimization version, MIN PIECE DISSECTION, is hard to approximate to within some constant ratio:

**Theorem 2.** *There is a constant* $\varepsilon_{\text{MPD}} > 0$ *such that it is NP-hard to approximate* MIN PIECE DISSECTION *to within a factor of* $1 + \varepsilon_{\text{MPD}}$ *of optimal.*[4]

Both results are based on essentially the same reduction, from 5-PARTITION for Theorem 1 or from MAX 5-PARTITION for Theorem 2. We present the common reduction in Section 4. We then prove Theorem 1 and Theorem 2 in Sections 5 and 6 respectively.

Restricting the kinds of polygons given as input, the kinds of cuts allowed, and the ways the pieces can be packed gives rise to many variant problems. Section 7 explains for which variants our results continue to hold.

[4] The best $\varepsilon_{\text{MPD}}$ we can achieve is $1/1080 - \varepsilon$ for any $\varepsilon \in (0, 1/1080)$.

## 4 The Reduction

This section describes a polynomial-time reduction from 5-PARTITION to $k$-PIECE DISSECTION and states a lemma crucial to both of our main proofs later in the paper. The proof of the lemma is deferred to the full version.

*Reduction from* 5-PARTITION *to* $k$-PIECE DISSECTION. Let $A = \{a_1, \ldots, a_n\}$ be the given 5-PARTITION instance and let $p = \frac{5}{n} \Sigma_{a \in A} a$ denote the target sum. Let $d_s = 12(\max_{a \in A} a + p)$ and $d_t = (n-1)d_s + \Sigma_{a \in A} a + 2 \max_{a \in A} a$. We create a source polygon $P$ consisting of *element rectangles* of width $a_i$ and height 1 for each $a_i \in A$ spaced $d_s$ apart, connected below by a rectangular *bar* of width $\Sigma_{a \in A} a + (\frac{n}{5} - 1)d_t$ and height $\delta = \frac{1}{10 \Sigma_{a \in A} a + 2(\frac{n}{5} - 1)d_t}$. The first element rectangle's left edge is flush with the left edge of the bar; the bar extends beyond the last element rectangle. Our target polygon $Q$ consists of $\frac{n}{5}$ *partition rectangles* of width $p$ and height 1 spaced $d_t$ apart, connected by a bar of the same dimensions as the source polygon's bar. The first partition rectangle's left edge and last partition rectangle's right edge are flush with the ends of the bar. The illustration of both polygons are given in Figure 1. Both polygons' bars have the same area and the total area of the element rectangles equals the total area of the partition rectangles, so the polygons have the same area. Finally, let the number of pieces $k$ be $n$.

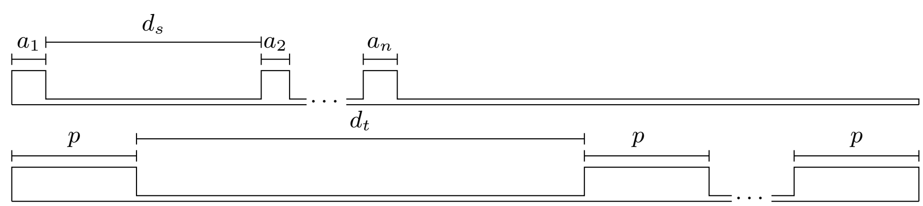


**Fig. 1.** The source polygon $P$ (above) and the target polygon $Q$ (below) are shown (not to scale). Length $d_t$ is longer than the distance between the leftmost edge of the leftmost element rectangle and the rightmost edge of the rightmost element rectangle.

*Reduction from* MAX 5-PARTITION *to* MIN PIECE DISSECTION. The optimization problem uses the same reduction as the decision problem, except that we do not specify $k$ for the optimization problem.

The idea behind our reduction is to force any valid dissection to cut each element rectangle off the bar in its own piece.[5] When $\delta$ is small enough, the resulting packing problem is a direct simulation of 5-PARTITION.

[5] Because $k = n$, $a_1$ will remain attached to the bar, forcing it to be the first element rectangle placed in the first partition rectangle. Because the order of and within partitions does not matter, this constraint does not affect the 5-PARTITION simulation.

Intuitively, each dissected piece should contain only one element rectangle. Our reduction sets $d_s$ large enough that any piece containing parts of two element rectangles does not fit in a partition rectangle. At the same time, we pick $d_t$ large enough that no piece can be placed in more than one partition rectangle. Thus one could plausibly prove that each element rectangle must be in its own piece.

Unfortunately, we were unable to prove that each element rectangle must be in its own piece. For each element rectangle, we define the *trimmed element rectangle* corresponding to each element rectangle as the rectangle resulting from ignoring the lower $4\delta$ of the element rectangle's height; see Figure 2. In other words, for each $a_i$, the corresponding trimmed element rectangle is the rectangle that shares the upper left corner with the element rectangle and is of width $a_i$ and height $1 - 4\delta$.

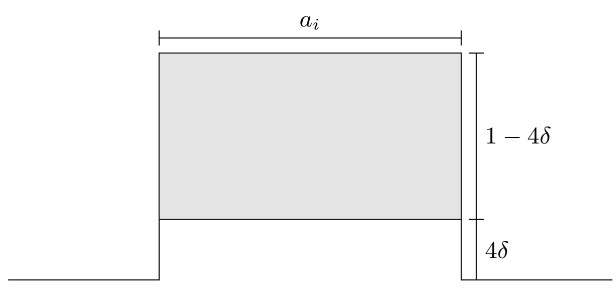


**Fig. 2.** The $i$th trimmed element rectangle.

While we could not prove that each element rectangle is in its own piece, we can prove the corresponding statement about trimmed element rectangles:

**Lemma 1.** *If $P$ can be cut into pieces that can be packed into $Q$, then each of these pieces intersect with at most one trimmed element rectangle.*

The proofs of both of our main theorems use this lemma. The intuition behind the proof of this lemma is similar to the intuitive argument for why each element rectangle should be in its own piece. As the details of the proof are not central to this paper, we defer the proof of this lemma to the full version.

## 5 Proof of NP-hardness of *k*-Piece Dissection

Before we prove Theorem 1, we state the result from [10] for 5-Partition:

**Theorem 3 ([10]).** 5-Partition *is NP-hard.*[6]

We now prove Theorem 1.

[6] As stated earlier, the result from [10] is for 4-Partition, but 4-Partition is easily reduced to 5-Partition; see Section 2.

*Proof (of Theorem 1).* We prove that the reduction described in the previous section is indeed a valid reduction from 5-PARTITION. The reduction clearly runs in polynomial time. We are left to prove that the instance of $k$-PIECE DISSECTION produced by the reduction is a YES instance if and only if the input 5-PARTITION is also a YES instance.

*(5-PARTITION $\Longrightarrow$ $k$-PIECE DISSECTION)* Suppose that the 5-PARTITION instance is a YES instance. Given a 5-PARTITION solution, we can cut all but the first element rectangle off the bar and pack them in the partition rectangles according to the 5-PARTITION solution. The piece containing the first element rectangle must be placed at the very left of the first partition rectangle, but we can reorder the partitions in the 5-PARTITION solution so that the first element is in the first partition. As a result, the $k$-PIECE DISSECTION instance is also a YES instance.

*($k$-PIECE DISSECTION $\Longrightarrow$ 5-PARTITION)* Suppose that the $k$-PIECE DISSECTION instance is a YES instance, i.e., $P$ can be cut into $k$ pieces that can then be placed into $Q$. By Lemma 1, no two trimmed element rectangles are in the same piece. Because there are $n = k$ such rectangles, each piece contains exactly one whole trimmed element rectangle. Because these pieces can be packed into $Q$, we must also be able to pack all the trimmed element rectangles into $Q$ (with some space in $Q$ left over).

Let $B_i$ be the set of all trimmed element rectangles (in the packing configuration) that intersect the $i$th partition rectangle. From our choice of $d_t$, each trimmed element rectangle can intersect with at most one partition rectangle. Moreover, no trimmed element rectangles fit entirely in the bar area, so each of them must intersect with at least one partition rectangle. This means that $B_1, \ldots, B_{n/5}$ is a partition of the set of all trimmed element rectangles. Let $A_i$ be the set of all integers in $A$ corresponding to the trimmed element rectangles in $B_i$. Observe that $A_1, \ldots, A_{n/5}$ is a partition of $A$.

We claim that $A_1, \ldots, A_{n/5}$ is indeed a solution for 5-PARTITION. Assume for the sake of contradiction that $A_1, \ldots, A_{n/5}$ is not a solution, that is, $\sum_{a \in A_i} a \neq p$ for some $i$. Because $\sum_{a \in A} a = p(n/5)$, there exists $j$ such that $\sum_{a \in A_j} a > p$. Because all $a \in A$ are integers and $p$ is an integer, $\sum_{a \in A_j} a \geq p + 1$.

Consider the $j$th partition rectangle. Define the *extended partition rectangle* as the area that includes a partition rectangle, the bar area directly below it, and the bar $\delta/2$ to the left and to the right of the partition rectangle. Figure 3 shows an extended partition rectangle enclosed in thick edges. (Ignore the shaded rectangle for the moment.)

Consider any trimmed element rectangle in the packing configuration that intersects with this partition rectangle. We claim that each such trimmed element rectangle must be wholly contained in the extended partition rectangle.

Consider the area of the trimmed element rectangle outside the partition rectangle and the bar below it. If this is not empty, this must be a right triangle with hypotenuse on the extension down to the bar of a vertical side of the partition rectangle (see Figure 3). The hypotenuse of this triangle is of length at

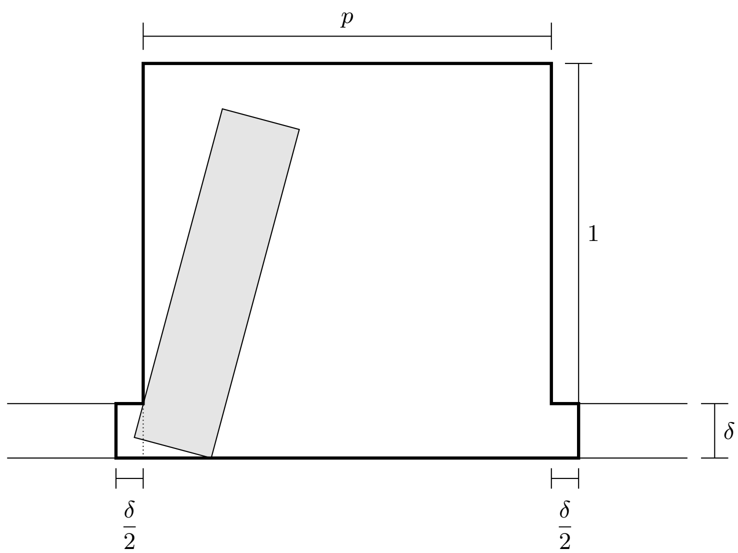


**Fig. 3.** The area enclosed by thick edges is the extended partition rectangle corresponding to this partition rectangle. In this configuration, the trimmed element rectangle, shown as the shaded area, is partially outside of the partition rectangle and the bar below it. This external area is a right triangle with hypotenuse on the extension of a vertical edge of the partition rectangle (shown as the dotted line segment), which is of length $\delta$.

most $\delta$, so the height of the triangle (perpendicular to the hypotenuse) is at most $\delta/2$. Thus, the triangle must be in the extended partition rectangle. Thus the whole trimmed element rectangle must be in the extended partition rectangle, as claimed.

The area of the extended partition rectangle is $p + p\delta + \delta^2 < p + 1/2$. However, the total area of the trimmed element rectangles contained in this area is $\sum_{a \in A_j} a(1 - 4\delta) = \sum_{a \in A_j} a - 4\delta \sum_{a \in A_j} a \geq (p + 1) - 4\delta \sum_{a \in A_j} a > p + 1/2$, which is a contradiction.

Thus we conclude that $A_1, \ldots, A_{n/5}$ is a solution to 5-Partition, which implies that the 5-Partition instance is a yes instance as desired. □

## 6 Proof of Inapproximability of Min Piece Dissection

In this section, we show the inapproximability of Min Piece Dissection via a reduction from the intermediate problem Max 5-Partition, whose inapproximability result is described in the following lemma.

**Lemma 2.** *There is a constant* $\alpha_{\mathrm{M5P}} > 1$ *such that* $\textsc{Gap}_{\textsc{Max-5-Partition}}[n(1 - \varepsilon)/5, n(1/\alpha_{\mathrm{M5P}} + \varepsilon)/5]$ *is NP-hard for any sufficiently small constant* $\varepsilon > 0$.[7]

[7] The best $\alpha_{\mathrm{M5P}}$ we can achieve here is $216/215$.

Lemma 2 implies that it is hard to approximate Max 5-Partition to within an $\alpha_{\text{M5P}} - \varepsilon$ ratio for any sufficiently small $\varepsilon > 0$. The proof of Lemma 2 largely relies on the reduction used to prove NP-hardness of 4-Partition in [10], but we apply our modified reduction on the inapproximability result of 4-Uniform 4-Dimensional Matching by Hazan, Safra, and Schwartz [12]. We defer the proof of this lemma to the full version. Here we focus on the gap preservation of the reduction, which implies Theorem 2.

**Lemma 3.** *There is a constant $\alpha_{\text{MPD}} > 1$ such that the following properties hold for the reduction described in Section 4:*

- *if the optimum of the* Max 5-Partition *instance is at least $n(1-\varepsilon)/5$, then the optimum of the resulting* Min Piece Dissection *instance is at most $n(1+\varepsilon/5)$; and*
- *if the optimum of the* Max 5-Partition *instance is at most $n(1/\alpha_{\text{M5P}} + \varepsilon)/5$, then the optimum of the resulting* Min Piece Dissection *is at least $n(\alpha_{\text{MPD}} + \varepsilon/5)$.*

Because it is NP-hard to distinguish the two cases of the input Max 5-Partition instance, it is also NP-hard to approximate Min Piece Dissection to within an $\alpha_{\text{MPD}} - \varepsilon$ ratio for any sufficiently small constant $\varepsilon > 0$. Thus, Lemma 3 immediately implies Theorem 2. It remains to prove Lemma 3:

*Proof (of Lemma 3).* We will show that both properties are true when we choose $\alpha_{\text{MPD}}$ to be $1 + (1 - 1/\alpha_{\text{M5P}})/5$.

*(*Max 5-Partition $\Longrightarrow$ Min Piece Dissection*)* Suppose that the input Max 5-Partition instance has optimum at least $n(1-\varepsilon)/5$. Let $A_1, \dots, A_{m'}$ be the optimal partition where $m' \geq n(1-\varepsilon)/5$. We cut $P$ into pieces as follows:

1. First, we cut every element rectangle except the first one from the bar.
2. Next, let the indices of the elements in $A - (A_1 \cup A_2 \cup \cdots \cup A_{m'})$ be $i_1, \dots, i_l$ where $1 \leq i_1 < i_2 < \cdots < i_l \leq n$.
3. For each $i = 1, \dots, n/5 - m'$, let $j$ be the smallest index such that $a_{i_1} + \cdots + a_{i_j} \geq ip$. Cut the piece corresponding to $a_{i_j}$ vertically at position $ip - \left(a_{i_1} + \cdots + a_{i_{j-1}}\right)$ from the left. (If the intended cut position is already the right edge of the piece, then do nothing.)

To pack these pieces into $Q$, we arrange all pieces whose corresponding elements are in partitions in the optimal Max 5-Partition solution, then pack the remaining pieces into the remaining partition rectangles using the additional cuts made in step 3. We leave the piece containing the first element rectangle (and the bar) at its position in $P$, but this does not constrain our solution because the other pieces and the partitions can be freely reordered.

The number of cuts in step 1 is $n-1$ and in step 3 is at most $n/5 - m' \leq \varepsilon n/5$. Thus the total number of cuts is at most $n - 1 + \varepsilon n/5$, so the number of pieces is at most $1 + (n - 1 + \varepsilon n/5) = n(1 + \varepsilon/5)$ as desired.

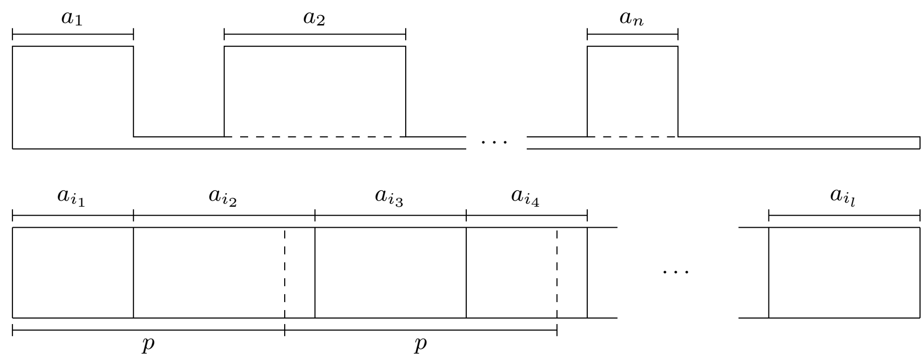


**Fig. 4.** An illustration of how the source polygon $P$ is cut. The cuts from step 1 are shown as dashed lines on the top figure; every element rectangle except the first one is cut from the bar. On the bottom, the cuts from step 3 are demonstrated. We can think of the cutting process as first arranging $a_{i_1}, \dots, a_{i_l}$ consecutively and then cutting at $p, 2p, \dots$.

*(*Min Piece Dissection $\implies$ Max 5-Partition*)* We prove this property in its contrapositive form. Suppose that the resulting Min Piece Dissection has an optimum of $k < n(\alpha_{\text{MPD}} + \varepsilon/5)$. Let us call these $k$ pieces $R_1, \dots, R_k$.

For each $i = 1, \dots, k$, let $R'_i$ denote the intersection between $R_i$ with the union of all trimmed element rectangles. By Lemma 1, each trimmed element rectangle can intersect with only one piece. This means that each $R'_i$ is a part of a trimmed element rectangle. (Note that $R'_i$ can be empty; in this case, we say that it belongs to the first trimmed element rectangle.)

Consider $R'_1, \dots, R'_k$. Because each of them is a part of a trimmed rectangle and there are $n$ trimmed rectangles, at most $k - n$ trimmed rectangles contain more than one of the $R'_i$. In other words, there are at least $n - (k - n) = 2n - k$ indices $i$ such that $R'_i$ is a whole trimmed element rectangle. Without loss of generality, suppose that $R'_1, \dots, R'_{2n-k}$ are entire trimmed element triangles.

We call a partition rectangle a *good partition rectangle* if it does not intersect with any of $R'_{2n-k+1}, \dots, R'_n$ in the packing configuration. From our choice of $d_t$, each $R'_i$ which is part of a trimmed element rectangle can intersect with at most one partition rectangle. As a result, there are at least $n/5 - (k - n)$ good partition rectangles.

For each good partition rectangle $O$, let $A_O$ be the subset of all elements of $A$ corresponding to $R'_i$s that intersect $O$. (Because $O$ is a good partition rectangle, each $R'_i$ that intersects $O$ is always a whole trimmed element rectangle.)

We claim that the collection of $T_O$'s for all good partition rectangles $O$ is a solution to the Max 5-Partition instance. We will show that this is indeed a valid solution. First, observe again that, because each $R'_i$ intersects with at most one partition rectangle, all $A_O$'s are mutually disjoint. Thus, we now only need to prove that the sum of elements of $A_O$ is exactly the target sum $p$.

Suppose for the sake of contradiction that there exists a good partition rectangle $O$ such that $\sum_{a \in A_O} a \neq p$. Consider the following two cases.

**Case 1:** $\sum_{a \in A_O} a > p$.

As we showed in the proof of Theorem 1, each trimmed element rectangle corresponding to $a \in A_O$ must be in the extended partition rectangle. By an argument similar to the argument used in the proof of Theorem 1, the total area of all these trimmed element rectangles is more than the area of the extended partition rectangle, which is a contradiction.

**Case 2:** $\sum_{a \in A_O} a < p$.

Because every $a \in A_O$ and $p$ are integers, $\sum_{a \in A_O} a + 1 \leq p$. From the definition of $A_O$, no trimmed element rectangles apart from those in $A_O$ intersect $O$. Hence the total area that trimmed element rectangles contribute to $O$ is at most

$$\left(\sum_{a \in A_O} a\right)(1 - 4\delta) < \sum_{a \in A_O} a \leq p - 1.$$

This means that an area of at least 1 unit square in $O$ is not covered by any of the trimmed element rectangles. However, the area of the source polygon outside of all the trimmed element rectangles is

$$\delta\left(\left(\frac{n}{5} - 1\right) d_t + \sum_{a \in A} a\right) + 4\delta\left(\sum_{a \in A} a\right) < 1,$$

which is a contradiction.

Hence, the solution defined above is a valid solution. Because the number of good partition rectangles is at least $n/5 - (k - n) > n/5 - n(\alpha_{\mathrm{MPD}} + \varepsilon/5 - 1) = n(1/\alpha_{\mathrm{M5P}} - \varepsilon)/5$, the solution contains more than $n(1/\alpha_{\mathrm{M5P}} - \varepsilon)/5$ subsets, which completes the proof of the second property. $\square$

## 7 Variations and Open Questions

Table 1 lists variations of $k$-Piece Dissection and whether our proofs of NP-hardness and inapproximability continue to hold. Because it is obvious from the proofs, we do not give detailed explanations as to why the proofs still work (or do not work) in these settings. Specifically:

1. Our proofs remain valid when the input polygons are restricted to be simple (hole-free) and orthogonal with all edges having integer length.[8]
2. Our results still hold under any cuts that leave each piece connected and Lebesgue measurable.
3. Our proofs work whether or not rotations and/or reflections are allowed when packing the pieces into $Q$.

**Table 1.** Variations on the dissection problem.

| Variation on | Variation description | Do our results hold? |
|---|---|---|
| Input Polygons | Polygons must be orthogonal | YES |
| | Polygons must be simple (hole-free) | YES |
| | Edges must be of integer length | YES |
| | Polygons must be convex | NO |
| Cuts Allowed | Cuts must be straight lines | YES |
| | Cuts must be orthogonal | YES |
| | Pieces must be simple (hole-free) | YES |
| | Pieces may be disconnected | NO |
| Packing Rules | Rotations are forbidden | YES |
| | Reflections are forbidden | YES |

While we have proved that the $k$-Piece Dissection is NP-hard and that its optimization counterpart is NP-hard to approximate, we are far from settling the complexity of these problems and their variations. We pose a few interesting remaining open questions:

- Is $k$-Piece Dissection in NP, or even decidable? We do not know the answer to this question even when only orthogonal cuts are allowed and rotations and reflections are forbidden. In particular, there exist two-piece orthogonal (staircase) dissections between pairs of rectangles which seem to require a cut comprised of arbitrarily many line segments [7, p. 60].
  If we require each piece to be a polygon with a polynomial number of sides, then problem becomes decidable. In fact, we can place this special case in the complexity class $\exists\mathbb{R}$, that is, deciding true sentences of the form $\exists x_1 : \cdots : \exists x_m : \varphi(x_1, \ldots, x_m)$ where $\varphi$ is a quantifier-free formula consisting of conjunctions of equalities and inequalities of real polynomials. To prove membership in $\exists\mathbb{R}$, use $x_1, \ldots, x_m$ to represent the coordinates of the pieces' vertices in $P$ and $Q$. Then, use $\varphi$ to verify that the pieces are well-defined partitions of $P$ and $Q$ and that each piece in $P$ is a transformation of a piece in $Q$; these conditions can be written as polynomial (in)equalities of degree at most two. $\exists\mathbb{R}$ is known to be in PSPACE [4].
- Is $k$-Piece Dissection still hard when one or both of the input polygons are required to be convex?
- Can we prove stronger hardness of approximation, or find an approximation algorithm, for Min Piece Dissection? The current best known algorithm for finding a dissection is a worst-case bound of a pseudopolynomial number of pieces [1].
- Is $k$-Piece Dissection solvable in polynomial time for constant $k$? Membership in FPT would be ideal, but even XP would be interesting.

**Acknowledgment.** We thank Greg Frederickson for helpful discussions.

[8] Our reduction uses rational lengths, but the polygons can be scaled up to use integer lengths while still being of polynomial size.

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