Straightening Polygonal Arcs and Convexifying Polygonal Cycles

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Abstract

Consider a planar linkage, consisting of disjoint polygonal arcs and cycles of rigid bars joined at incident endpoints (polygonal chains), with the property that no cycle surrounds another arc or cycle. We prove that the linkage can be continuously moved so that the arcs become straight, the cycles become convex, and no bars cross while preserving the bar lengths. Furthermore, our motion is piecewise-differentiable, does not decrease the distance between any pair of vertices, and preserves any symmetry present in the initial configuration. In particular, this result settles the well-studied carpenter's rule conjecture.

1. Introduction

Consider a finite embedded polygonal arc in the plane (by an *arc* we mean a homeomorphic image of the closed interval $[0, 1]$). Such a polygonal arc is often called an *open polygonal chain*. It has been an outstanding question as to whether it is possible to continuously move a polygonal arc in such a way that each edge remains a fixed length, there are no self-intersections during the motion, and at the end of the motion the arc lies on a straight line. This has come to be known as the carpenter's rule problem. A related question is whether it is possible to continuously move a polygonal simple closed curve in the plane, often called a *closed polygonal chain* or *polygon*, again without creating self-intersections or changing the length of the edges, so that it ends up a convex closed curve. We solve both problems here by showing that in both cases there is such a motion.

Physically, we think of a polygonal arc as a *linkage* or *framework* with hinges at its vertices, and rigid bars at its edges. The hinges can be folded as desired, but the bars must maintain their length and cannot cross. Motions of such linkages have been studied in discrete and computational geometry [3, 14, 19, 21, 25, 26, 28, 32, 34, 35, 42], in knot theory [7, 24], and in molecular biology and polymer physics [16, 22, 23, 24, 29, 30, 43]. Applications of this field include robotics, wire bending, hydraulic tube folding, and the study of macromolecule folding [26, 32].

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We say an arc is *straightened* by a motion if at the end of the motion it lies on a straight line. We say a polygonal simple closed curve (or *cycle*) is *convexified* by a motion if at the end of the motion it is a convex closed curve. All motions must be *proper* in the sense that no self-intersections are created, and each edge length is kept fixed. It is easy to see that if any cycle can be convexified by a motion, then any arc can be straightened by a motion: simply extend each arc to a cycle and convexify it. It is then easy to straighten the portion of the cycle that is the original arc.

It seems intuitively easy to straighten an entangled chain: just grab the ends and pull them apart. Similarly, a cycle might be opened by blowing air into it until it expands. But these methods have the difficulty that they may introduce singularities, where the arc or cycle may intersect itself. Our approach is to use an *expansive* motion in which all distances between two points increase. We also show that the area of a polygon increases in such an expansive motion.

We consider the more general situation, which we call an *arc-and-cycle set* A, consisting of a finite number of polygonal arcs and polygonal simple closed curves in the plane, with none of the arcs or cycles intersecting each other or having self-intersections. We say that A is in an *outer-convex configuration* if each component of A that is not contained in any cycle of A is either straight (when it is an arc) or convex (when it is a cycle).

We say that a motion of an arc-and-cycle set A is *expansive* if for every pair of vertices of A the distance increases or stays the same during the motion. We say that the motion is *strictly expansive* if in addition, for those vertices not on a straight subarc in a component of A and not on or in a common convex cycle of A, the distance between them increases strictly. We say that A is *separated*, if there is a line L in the plane such that L is disjoint from A and at least one component of A lies on each side of L.

Our main result is the following.

Theorem 1 *Every arc-and-cycle set has a piecewise-differentiable proper motion to an outer-convex configuration. Moreover, the motion is strictly expansive until the arc-and-cycle set is separated.*

We can also insist that the motion be strictly expansive during the entire motion, but this involves a great deal more complexity in the proof; see the full paper. Note that when there is just one component in the arc-and-cycle set, there is no difference between Theorem 1 and this extension.

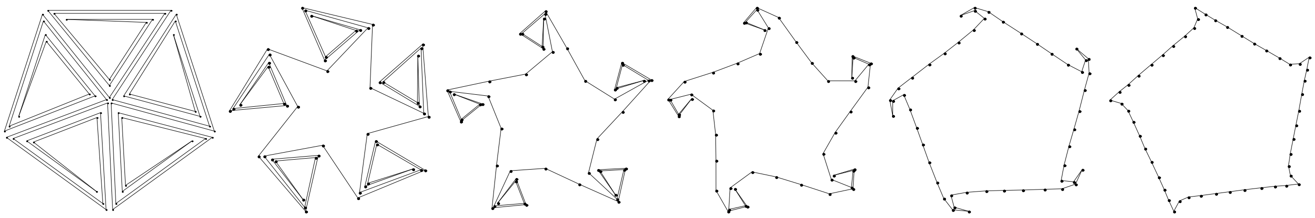

Figure 1. Convexifying a polygon that comes from doubling each edge in a locked tree. Snapshots are zoomed different amounts to improve visibility; each edge stays the same length throughout the motion. See `http://daisy.uwaterloo.ca/~eddemain/linkage` **for more animations.**

In contrast to this result, in dimension three there are arcs that cannot be straightened and polygons that cannot be convexified [3, 7]. In dimension four, all arcs and cycles *unlock*, i.e., can be straightened and convexified, respectively [8]. In the plane, there are examples of trees embedded in the plane that are *locked* in the sense that they cannot be properly moved so that the vertices lie nearly on a line [4]. In other words, there are two embeddings of the tree such that there is no proper motion from one configuration to the other. The important difference between trees and arc-and-cycle sets is that arc-and-cycle sets have maximum degree two.

	Arcs and Cycles	Trees
2-D	Not lockable (this paper)	Lockable [4]
3-D	Lockable [3, 7]	Lockable
4-D$^+$	Not lockable [8]	Not lockable?

Table 1. Summary of what types of linkages can be locked. The question mark denotes a conjecture.

Whether every arc in the plane can be straightened, and whether every polygon in the plane can be convexified, have been outstanding open questions until now. The problems are natural, so they have arisen independently in a variety of fields, including topology, pattern recognition, and discrete geometry. We are probably not aware of all contexts in which the problem has appeared. To our knowledge, Stephen Schanuel first invented the problem of convexifying cycles in the early 1970's, and George Bergman suggested the simpler question of straightening arcs. Ulf Grenander posed the problems during a talk in March 1987, and possibly earlier [personal communication with Alan Edmonds]. In the discrete and computational community, the problems were independently posed by William Lenhart and Sue Whitesides in March 1991 and by Joseph Mitchell in December 1992.

Solutions were already known for the special cases of monotone cycles [5] and star-shaped cycles [15], and for certain types of "externally visible" arcs [6].

A fairly large group of people, mentioned in the acknowledgments, was involved in trying to construct and prove or disprove locked arcs and cycles, at various times over the past few years. Typically, someone in the group would distribute an example that s/he constructed or was given by a colleague. We would try various motions that did not work, and we would often try proving that the example was locked because it appeared so! For some examples, it took several months before we found an unlocking motion. The main difficulty was that "simple" motions that change a few vertex angles at once, while easiest to visualize, seemed to be insufficient for unlocking complex examples. Amazingly, it also seemed that nevertheless there was always a global unlocking motion, and furthermore it was felt that there was a driving principle permitting "blowing up" of the linkage. This notion was formalized by third author with the idea that perhaps an arc could be straightened via an expansive motion.

The new tools that are applied here come from the theory of mechanisms and rigid frameworks. Arcs and cycles can be regarded as frameworks. See [1, 2, 9, 10, 11, 12, 13, 18, 27, 37, 38, 39, 40, 41] for relevant information about this theory.

Our approach is to prove that for any configuration there is an infinitesimal motion that increases all distances. Because of the nature of the arc-and-cycle set, this implies that there is a motion that works at least for a small expansive perturbation. We then combine these local motions into one complete motion. These notions are described in the rest of this paper. Section 3 proves the existence of infinitesimal motions using the nonexistence of certain stresses, a notion dual to infinitesimal motions for the underlying framework. The analysis of these stresses uses a lifting theorem from the theory of rigidity that was known to James Clerk Maxwell and Luigi Cremona [11, 12, 36] in the nineteenth century. Section 4 shows how to maneuver through the space of local motions to find a global motion with the desired properties.

2. Basics

A *linkage* or *bar framework* $G(\mathbf{p})$ is a finite graph $G = (V, E)$ without loops or multiple edges, together with a corresponding configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of n points in the plane, where $\mathbf{p}_i$ corresponds to vertex $i \in V$. (For con-

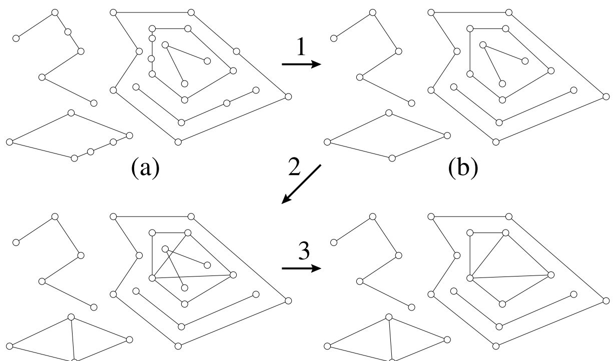

Figure 2. (a) Original arc-and-cycle framework. (b) With straight vertices removed. (c) With convex cycles rigidified. (d) With components nested within convex cycles removed.

venience we assume $V = \{1, \ldots, n\}$.) The edges of G constitute the set E and correspond to the bars in the framework, i.e., the links of a linkage. Arc-and-cycle sets are a particular kind of bar framework in which the graph G is a disjoint union of paths and cycles.

A *flex* or *motion* of $G(\mathbf{p})$ is a set of continuous functions $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$, defined for $0 \le t \le 1$, such that $\mathbf{p}(0) = \mathbf{p}$ and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is constant for each $\{i, j\} \in E$. We are interested in finding a motion of the arc-and-cycle set with the additional property that it is strictly expansive.

We begin with a basic property of expansive motions.

Lemma 1 *Any expansive motion of an arc-and-cycle set only increases the distance between two points on the arc-and-cycle set (each either a vertex or on a bar). In particular, there can be no self-intersections.*

2.1. The Framework $G_A(\mathbf{p})$

Given an arc-and-cycle set A that we would like to move to an outer-convex configuration, we make four modifications to A. The first three modifications simplify the problem by removing a few special cases that are easy to deal with; see Figure 2. The fourth modification will bring the problem of finding a strictly expansive motion into the area of tensegrity theory. In the end we will have defined a new framework, $G_A(\mathbf{p})$, which we will use throughout the rest of the proof.

Modification 1: Remove straight vertices. First we show that our arc-and-cycle set can be assumed to have no straight vertices, i.e., vertices with angle π. Furthermore, if during an expansive motion of the arc-and-cycle set we find that a vertex becomes straight, we can proceed by induction. For once the arc-and-cycle set has a straight subarc of more than one bar, we can coalesce this subarc into a single bar, thereby preserving the straightness of the subarc throughout the motion once it becomes straight. This reduces the number of bars and the number of vertices in the framework. By induction, this reduced framework has a motion according to Theorem 1, and such a motion extends directly to the original framework. The resulting motion is also strictly expansive by Lemma 1.

Modification 2: Rigidify convex polygons. Once a cycle becomes convex, we no longer have to expand it, and indeed we can hold it rigid from that point on. Of course, we allow a convex cycle to translate or rotate in the plane, but its vertex angles are not allowed to change. This can be directly modeled in the bar framework by introducing bars in addition to the arc-and-set cycle. Specifically, we add the edges of a triangulation of a cycle once that cycle becomes convex.

Modification 3: Remove components nested within convex cycles. The previous modification did not address the fact that components can be nested within cycles. Once a cycle becomes convex, not only can we rigidify it, but we can also rigidify any nested components, and treat them as moving in synchrony with the convex cycle. We do this by removing from the framework any components nested within a convex cycle. Assuming there were some nested components to deal with, this results in a framework with fewer vertices and fewer bars. By induction, this reduced framework has a motion according to Theorem 1. This motion can be extended to apply to the original framework by defining nested components to mimic the rigid motion of the containing convex cycle. It can be shown similar to Lemma 1 that the resulting motion is also strictly expansive.

Modification 4: Add struts. In order to model the expansive property we need, we apply the theory of tensegrity frameworks, in which frameworks can consist of both bars and "struts." In contrast to a bar which must stay the same length throughout a motion, a *strut* is permitted to increase in length, or stay the same length. Specifically, we add a strut between nearly every pair of vertices in the framework. The exceptions are those vertices already connected by a bar, and vertices on a common convex cycle, because in both cases we cannot hope to strictly increase the distance.

Final framework: $G_A(\mathbf{p})$. The above modifications define a tensegrity (bar-and-strut) framework $G_A(\mathbf{p})$ in terms of the arc-and-cycle set A. Specifically, assume that A has no straight vertices or components nested within convex components. We call such an arc-and-cycle set *reduced*. We define the set B of bars by starting with the set of bars from the arc-and-cycle set, and adding a triangulation of every convex cycle. The set S of struts consists of all vertex pairs which are not in B and which do not belong to the same convex cycle. See Figure 3 for an example.

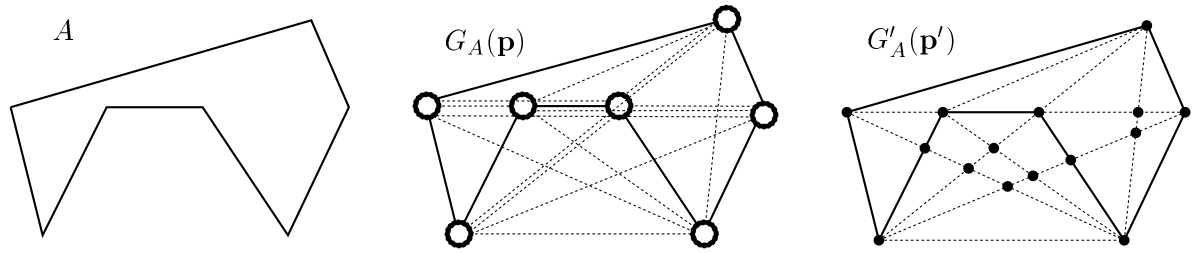

Figure 3. Construction of the frameworks $G_A(\mathbf{p})$ and $G'_A(\mathbf{p}')$. Solid lines denote bars, and dashed lines denote struts.

Our goal in the proof of Theorem 1 is to find a motion such that all bars maintain their length, while all struts strictly increase in length, in other words, a motion of $G_A(\mathbf{p})$ that is *strict* on all struts.

Thus, we want to find a motion $\mathbf{p}(t)$ for $0 \le t \le 1$ such that $\mathbf{p}(0) = \mathbf{p}$ and

$$\frac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| = 0 \quad \text{for } \{i,j\} \in B,$$
$$\frac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| > 0 \quad \text{for } \{i,j\} \in S.$$

Differentiating the *squared* distances $\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|^2 = (\mathbf{p}_j(t) - \mathbf{p}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t))$ and denoting the velocity vectors by $\mathbf{v}_i(t) := \frac{d}{dt}\mathbf{p}_i(t)$, we obtain the following equivalent conditions.

$$(\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) = 0 \quad \text{for } \{i,j\} \in B,$$
$$(\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) > 0 \quad \text{for } \{i,j\} \in S.$$

Intuitively, the first-order change in the distance between vertex i and j is modeled by projecting the velocity vectors onto the line segment between the two vertices; see Figure 4.

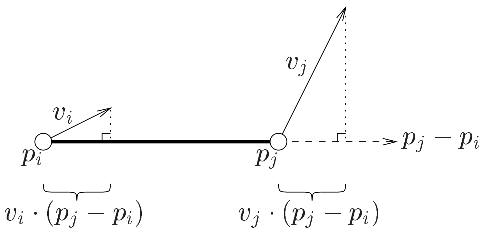

Figure 4. The dot product $(\mathbf{v}_j(t)-\mathbf{v}_i(t))\cdot(\mathbf{p}_j(t)-\mathbf{p}_i(t))$ is zero if the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ stays the same to the first order, positive if the distance increases, and negative if the distance decreases.

2.2. Infinitesimal Motions

A *strict infinitesimal motion* or *strict infinitesimal flex* $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ specifies the first derivative of a strictly expansive motion at time 0. In other words, it assigns a velocity vector $\mathbf{v}_i$ to each vertex i so that it preserves the length of the bars to the first order, and strictly increases the length of struts to the first order. More precisely, a strict infinitesimal motion must satisfy

$$\begin{aligned}(\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i,j\} \in B,\\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) > 0 \text{ for } \{i,j\} \in S,\end{aligned} \qquad (1)$$

where $\mathbf{p}_i$ denotes the initial position of vertex i.

In the next section, we prove that such a strict infinitesimal motion always exists. In Section 4 we show how this leads to motions for small amounts of time. These motions are then shown to continue globally until the configuration reaches an outer-convex configuration.

3. Local Motion

Recall that an arc-and-cycle set is called *reduced* if adjacent collinear bars have been coalesced, and components nested within cycles have been removed. In this section, we prove the following:

Theorem 2 *For any reduced arc-and-cycle set A there is an infinitesimal flex $\mathbf{v}$ of the corresponding bar-and-strut framework $G_A(\mathbf{p})$ satisfying* (1).

3.1. Equilibrium Stresses

The equations and inequalities in (1) form a linear feasibility problem that is common for tensegrity frameworks. But in order to solve this problem it is helpful to restate it in terms of the dual problem. This leads to the study of equilibrium stresses in tensegrity frameworks.

A *stress* in a framework $G(\mathbf{p})$ is an assignment of a scalar $\omega_{i,j} = \omega_{j,i}$ to each edge $\{i,j\}$ of G (a bar or strut). The whole stress is denoted by $\omega = (\ldots, \omega_{i,j}, \ldots)$. We say that the stress ω is an *equilibrium stress* if for each vertex i of G the following equilibrium equation holds:

$$\sum_{j\,:\,\{i,j\}\in B\cup S} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0 \qquad (2)$$

We say that the stress ω is *proper* if furthermore for all struts $\{i,j\}$, $\omega_{i,j} \ge 0$. There is no sign condition for bars. Now we state the duality between equilibrium stresses and infinitesimal motions.

Theorem 3 *The framework $G_A(\mathbf{p})$ corresponding to a reduced arc-and-cycle set A has only the zero proper equilibrium stress.*

Lemma 2 *Theorem 3 implies Theorem 2.*

This equivalence is a standard technique in the theory of rigidity. It is proved in [10, Theorem 2.3.2]. See also [27, Theorem 10] for a similar result. It can also be proved using linear programming duality.

3.2. Planarization

To prove that only the zero equilibrium stress exists (i.e., to prove Theorem 3), we use another tool in rigidity called the Maxwell-Cremona theorem. Before we can apply this tool, we need to transform the framework $G_A(\mathbf{p})$ into a planar framework $G'_A(\mathbf{p}')$. (Refer to the framework on the right of Figure 3.) We introduce new vertices at all intersection points between edges of $G_A(\mathbf{p})$, and subdivide the bars and struts accordingly. Any multiple edges resulting from this operation are merged. We define the resulting framework $G'_A(\mathbf{p}')$ to have bars precisely covering the bars of $G_A(\mathbf{p})$. All the other edges of $G'_A(\mathbf{p}')$ are struts. $G'_A(\mathbf{p}')$ is *planar* in the sense that two edges (bars or struts) intersect only at a common endpoint.

Despite the added points in this modification, the planar framework $G'_A(\mathbf{p}')$ is equivalent to the original framework $G_A(\mathbf{p})$ in the sense of equilibrium stresses. Indeed, the following stronger statement holds. Call a stress *outer-zero* if the only edges that carry a nonzero stress are the edges of convex cycles and the edges interior to convex cycles. A stress is *outer-nonzero* if it is nonzero on some edge that is exterior to all convex cycles and is not an edge of any convex cycle.

Lemma 3 *If $G_A(\mathbf{p})$ has a proper equilibrium stress ω, then $G'_A(\mathbf{p}')$ has an outer-nonzero proper equilibrium stress ω'.*

Proof: During the modifications to $G_A(\mathbf{p})$ that made $G'_A(\mathbf{p}')$, we modify ω to make ω' as follows. When we subdivide an edge $\{i,j\}$ with stress $\omega_{i,j}$, each edge of the subdivision $\{k,l\}$ gets the stress $\omega_{i,j}||p_i - p_j||/||p_k - p_l||$. (The ratio of lengths is necessary because $\omega_{i,j}$ is a weight, and the actual force comes from scaling by the length of the edge $\{i,j\}$; see (2).) When merging several edges, we add the corresponding stresses. It is easy to verify that the resulting stress is proper and in equilibrium. We only have to check that positive and negative stresses do not completely cancel during the merging process, and that the stress is furthermore outer-nonzero.

First note that some strut $\{i,j\}$ of $G_A(\mathbf{p})$ carries a positive stress. In other words, $G_A(\mathbf{p})$ cannot be stressed only on its bars; in particular, arcs, cycles, and triangulated convex cycles cannot carry a nonzero stress. This follows because, in any such bar framework, there is a degree-two vertex v; in particular, every triangulated convex cycle has a degree-two vertex (an ear). Because the framework is reduced, the two bars incident to v lie in a strictly convex wedge at the vertex, so these two bars cannot carry stress while satisfying equilibrium at v. Any nonzero stress is thus nonzero on the rest of the framework, but by induction, this cannot occur. Hence, the bar framework cannot carry a nonzero stress, so some strut must have a nonzero stress.

The conditions of Theorem 2 enforce that no angles at vertices of the arc-and-cycle set are π or 0: an angle of π would create a straight subarc of two bars (contradicting the assumption that framework is reduced), and an angle of 0 would violate simplicity. Thus, no strut of $G_A(\mathbf{p})$ is completely covered by bars. Therefore, for the strut $\{i,j\}$ of $G_A(\mathbf{p})$ that carries a positive stress, some portion of it in $G'_A(\mathbf{p}')$ will also have a positive stress, because a positive stress can only be canceled by a stress on a bar. In particular, ω' must be nonzero.

Furthermore, if the strut $\{i,j\}$ is exterior to all convex cycles in A, we have that ω' is outer-nonzero. Now suppose that $\{i,j\}$ is partially interior to convex cycles in A (by construction, the strut cannot be entirely within convex cycles of A). Then there is a portion of $\{i,j\}$ with the property that it is incident to a convex cycle and exterior to all convex cycles in A. This portion must be uncovered by bars, because no bar in A has this property, and the only additional bars in $G_A(\mathbf{p})$ are interior to convex cycles. Hence, the corresponding strut in $G'_A(\mathbf{p})$ carries a positive stress, so ω' is outer-nonzero in all cases. □

Thus, to prove that the original framework $G_A(\mathbf{p})$ has only the zero proper equilibrium stress, it suffices to prove that the planar framework $G'_A(\mathbf{p}')$ has only outer-zero proper equilibrium stresses.

3.3. Maxwell-Cremona Theorem

To prove that only outer-zero equilibrium stresses exist, we employ the Maxwell-Cremona correspondence between equilibrium stresses in planar frameworks and three-dimensional polyhedral graphs that project onto these frameworks. More precisely, a *polyhedral graph* or *polyhedral terrain* Γ comes from lifting a planar framework into three dimensions—that is, assigning a z coordinate (positive or negative) to each vertex in the framework—such that each face bounded by edges of the framework (including the exterior face) remains planar. The polyhedral surface Γ is then the graph of a piecewise-linear continuous function of two variables that is linear on the faces determined by $G'_A(\mathbf{p}')$.

Consider an edge $\{i,j\}$ in a planar framework, separating faces F and F'. Let $z = \mathbf{a}\cdot\mathbf{p} + b$ and $z = \mathbf{a}'\cdot\mathbf{p} + b'$ be the two linear functions specifying Γ on F and F', respectively. A straightforward calculation reveals that the vector $\mathbf{a}' - \mathbf{a}$ must be perpendicular to the edge $\{i,j\}$:

$$\mathbf{a}' - \mathbf{a} = \omega_{i,j}\mathbf{e}^{\perp}_{i,j} \tag{3}$$

where $\mathbf{e}^{\perp}_{i,j}$ is a vector of the same length as the vector $\mathbf{p}_j - \mathbf{p}_i$, perpendicular to it, and pointing from F towards F'. We call the edge $\{i,j\}$ a *valley* if $\omega_{i,j} > 0$, a *mountain* if $\omega_{i,j} < 0$, and *flat* if $\omega_{i,j} = 0$. Note that the two sides of a valley do not necessary "go up" in z (and so a valley might not carry water); however, as one crosses a valley, the slope gets steeper. A similar remark applies to mountains.

Theorem 4 (Maxwell-Cremona Theorem) *(i) For every polyhedral graph Γ that projects to a planar bar frame-*

work $G(\mathbf{p})$, the stress ω defined by (3) forms an equilibrium stress on $G(\mathbf{p})$.

(ii) For every proper equilibrium stress ω in a planar framework $G(\mathbf{p})$, $G(\mathbf{p})$ can be lifted into a polyhedral graph Γ such that (3) holds for all edges. In particular, edges with positive stress lift to valleys, edges with negative stress lift to mountains, and edges with no stress lift to flat edges. Furthermore, Γ is unique up to addition of affine-linear functions.

A proof of this result can be found in [13, 20, 36], which follows the idea suggested above. Another point of view [17] is that the stresses are a scaling of the angular momentum vectors of the function that lifts from the plane to the graph.

3.4. Main Argument

Note in particular that the zero equilibrium stress corresponds to the trivial polyhedral graph in which all faces are coplanar (i.e., defined by the same linear function). More generally, an outer-zero equilibrium stress corresponds to a polyhedral graph that is flat on every edge exterior to all convex cycles. Therefore, to prove that only outer-zero equilibrium stresses exist, and hence prove Theorem 3, it suffices to show that only such polyhedral graphs exist.

More precisely, consider any polyhedral graph Γ that projects to the planar framework $G'_A(\mathbf{p}')$ with the property that all struts are lifted to valleys or flat edges (because they can only carry nonnegative stress), and bars are lifted to valleys, mountains, or flat edges. We need to show that nonflat edges can only appear within or on the boundary of convex cycles. Because we may add an arbitrary affine-linear function to a graph, we may conveniently assume that the exterior face of Γ is on the xy plane. Thus the problem is to show that Γ does not lift off the xy plane any vertex of $G'_A(\mathbf{p}')$ except possibly vertices interior to convex cycles.

One simple fact that we will need is the following:

Lemma 4 *Any mountain in the polyhedral graph Γ projects to a bar in the planar framework $G'_A(\mathbf{p}')$.*

Proof: A strut can only carry nonnegative stress, so by Theorem 4 it can only lift to a valley or a flat edge. □

We now come to the heart of our proof, the proof of Theorem 5. It is here we finally show that the stress must be outer-zero, by looking at the maximum of any Maxwell-Cremona lift. Specifically, let M denote the region in the xy plane where the z value attains its maximum in Γ, which is a nonempty union of faces, edges, and vertices of the planar framework $G'_A(\mathbf{p}')$. The following statement immediately implies Theorem 3 and hence Theorem 2:

Theorem 5 *The set M includes every face of the framework $G'_A(\mathbf{p}')$ that is exterior to all convex cycles.*

Consider the boundary ∂M, which may be empty if M fills the whole plane. Because points in M lift to maximum height, all edges of ∂M must lift to mountains. Thus by Lemma 4, all edges of ∂M must be bars in the framework. Hence, ∂M consists of disjoint vertices, paths of edges, and complete cycles of the arc-and-cycle set, together with a subset of the triangulations of the convex components. Figure 5 shows the main possibilities. We will show that the only case in Figure 5 that can actually occur is (l), in which ∂M includes a convex cycle and M includes the local exterior of that cycle.

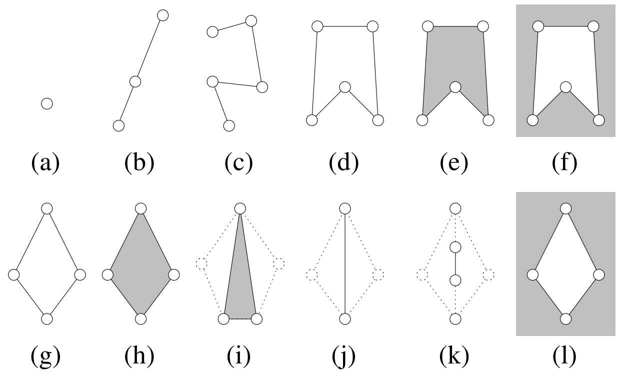

Figure 5. Hypothetical connected components of ∂M and their relation to M. Solid lines are edges of ∂M; white regions are absent from M; and shaded regions are present in M. (a) An isolated vertex. (b) A straight subarc. (c) A nonstraight subarc. (d) A nonconvex cycle. (e) A nonconvex cycle and its local interior. (f) A nonconvex cycle and its local exterior. (g–k) Various situations with a convex cycle. (l) The only possible case: A convex cycle and its local exterior.

Our main technique for arriving at a contradiction in all cases except (l) is that of *slicing* the polyhedral graph. Consider a plane Π that is parallel to the xy plane and just below the maximum z coordinate of Γ. (By "just below" we mean that Π is above all vertices of Γ not at the maximum z coordinate.) Now take the intersection of Π with the surface Γ, and project this intersection to the xy plane. The resulting set X is shown in Figure 6 for the various cases.

The set X captures several properties of the polyhedral graph Γ. First note that because X is the boundary of a small neighborhood of M in the plane, it is a disjoint union of cycles. It is also polygonal. Each edge of X corresponds to a face of Γ, and each vertex of X corresponds to an edge of Γ. The *angle* at a vertex v of X (interior to the side bounded by M) determines the type of edge corresponding to v: the angle is π (straight) if the edge is flat, less than π (convex) if the edge is a mountain, and more than π (reflex) if the edge is a valley.

The basic idea is to show that X has "many" convex angles, and apply Lemma 4 to prove that the framework has

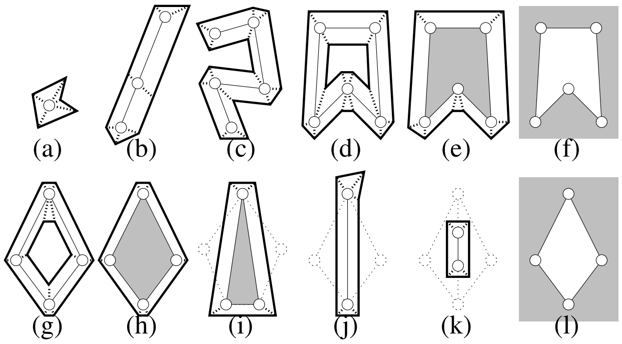

Figure 6. Slicing the polyhedral graph Γ just below the maximum z coordinate, in each case corresponding to those in Figure 5. Thick lines denote the slice intersection X, and thick dotted lines denote the corresponding edges in the polyhedral graph Γ.

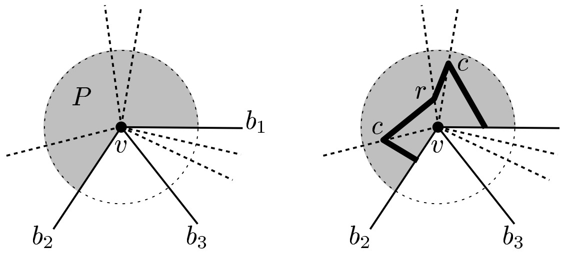

Figure 7. (Left) Illustration of Lemma 5: solid lines are bars, dotted lines are struts, and the shaded pie wedge P must be contained in M. (Right) Illustration of the proof; the thick lines form the portion of X inside P, and the symbols c and r denote convex and reflex vertices, respectively.

"too many" bars. The key fact underlying the proof is that the arc-and-cycle set has maximum bar-degree two: every vertex is incident to at most two bars. In the planar framework $G'_A(\mathbf{p}')$, only vertices v of convex cycles can have bar-degrees greater than two, and these bars are contained in a convex wedge from v.

Our proof deals with all cases at once. To illustrate the essence of the proof, we first describe it for the special case (a) in which one component of ∂M is a single vertex v that does not belong to a convex cycle. In this case, one component of X is a star-shaped polygonal cycle P around v. Every polygon P has at least three convex vertices. (Because the turn angles of a polygon sum to 2π, and the maximum turn angle of a vertex is $< \pi$, every polygon has at least three vertices with positive turn angles.) These three convex vertices correspond to three mountains in Γ, all incident to a common vertex v. By Lemma 4, there are three bars incident to v, contradicting the maximum-degree-two property for vertices not on convex cycles. Therefore, case (a) cannot exist.

The general reason that cases (a–k) cannot exist is the following:

Lemma 5 *Let v be a vertex on the boundary of M, and let $b_1, \ldots, b_k$ be the bars incident to v in cyclic order. Consider a small disk D around v.*

(a) If there is an angle of at least π at v between two consecutive bars, say b_i and b_{i+1}, then the pie wedge P of D bounded by b_i and b_{i+1} belongs to M. (See Figure 7.)

(b) If there are no bars or only one bar incident to v, i.e., $k \leq 1$, then the entire disk D belongs to M. (This can be viewed as a special case of (a).)

Proof: (a) Because there are no bars in the pie wedge P, and hence no edges of ∂M in P, P must be completely contained in or disjoint from M. Assume to the contrary that P is disjoint from M. Then the intersection of the slice X with the pie wedge P is a star-shaped polygonal arc around v starting from a point on b_i and ending at a point on b_{i+1}. By the properties of X, convex vertices on this arc correspond to mountains emanating from v, and reflex vertices correspond to valleys emanating from v. Because the angle of the pie wedge P is at least π, the arc must have at least one convex vertex in P. (The turn angles along the arc must sum to a positive number, so some vertex must have a positive turn angle.) By Lemma 4, there must be a bar in P, a contradiction.

(b) If $k = 1$, the bars b_i and b_{i+1} coincide, and the same proof applies. The star-shaped polygonal arc becomes a star-shaped polygonal cycle, which must have at least two convex vertices not on $b_i = b_{i+1}$. If $k = 0$, X also has a star-shaped polygonal cycle around v, which must have at least three convex vertices, yet v has no incident bars. $\square$

Note that this lemma applies to every vertex in our planar framework $G'_A(\mathbf{p}')$, because every vertex either has bar-degree at most two or is a vertex of a convex cycle, and in either case there is a nonconvex angle between two consecutive bars.

One can immediately verify that the examples shown in Figure 6(a–k) contradict Lemma 5. For example, applying the lemma to any vertex of ∂M shows that M should contain a positive two-dimensional area incident to that vertex. This immediately rules out cases (a–d), (g), (j), and (k).

A general proof is also easy with Lemma 5 in hand:

Proof (Theorem 5): Consider first a degree-0 or degree-1 vertex v in ∂M. (Such a point would appear when M has a component that is an isolated point or an arc of bars.) Because Lemma 5 applies to every vertex of the framework, we know that some positive two-dimensional area in the vicinity of v belongs to M, contradicting that v has degree 0 or 1 in ∂M. [This rules out cases (a–c) and (j–k).]

It follows that ∂M is a union of cycles. A component of ∂M can be of two kinds:

(i) If it is formed from the edges of a convex cycle and its triangulation, Lemma 5 applies to any vertex in it, and we conclude that M contains the face of the framework immediately exterior to the cycle. [This rules out cases (g–i).]

(ii) If it consists of a complete nonconvex cycle, we can apply Lemma 5 to some convex vertex and to some reflex vertex (they must both exist), and we conclude that M contains both the face of the framework immediately interior and the face immediately exterior to the cycle. [This rules out cases (d–f).]

In the end, the only faces of the framework that can be excluded from M are those interior to convex cycles [case (l)]. This completes the proof of Theorem 5 and of Theorems 3 and 2. □

4. Global Motion

In this section, we combine the infinitesimal motions into a global motion, thereby proving Theorem 1, the main theorem. In Theorem 2 we have established the existence of *some* direction of motion $\mathbf{v}$. Now we select a unique vector $\mathbf{v} := f(\mathbf{p})$ for each configuration $\mathbf{p}$ as the solution of a convex optimization problem (4–6). We then set up the differential equation

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)).$$

The solution of this differential equation moves the linkage to a configuration where an angle between two bars becomes straight. At this point we merge the two bars and continue with the reduced framework that has one vertex less. This procedure is iterated until the framework is outer-convex and no further expansive motion is possible.

It is convenient for the proof of Theorem 1 to effectively pin an edge in the configuration. Choose any edge, say $\{\mathbf{p}_1, \mathbf{p}_2\}$, that is a bar. During the motion we will arrange matters so that this bar is stationary.

We now go into the details of the proof. We use the following nonlinear minimization problem to define a unique direction $\mathbf{v}$ for every configuration $\mathbf{p}$ of a reduced arc-and-cycle set.

$$\text{minimize} \quad \sum_{i \in V} \|\mathbf{v}_i\|^2 + \sum_{\{i,j\} \in S} \left[(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|\right]^{-1} \quad (4)$$

$$\begin{aligned} \text{subject to} \quad (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &> \|\mathbf{p}_j - \mathbf{p}_i\|, \\ &\text{for } \{i,j\} \in S \quad (5) \\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &= 0, \\ &\text{for } \{i,j\} \in B \quad (6) \\ \mathbf{v}_1 = \mathbf{v}_2 &= 0 \quad (7) \end{aligned}$$

The restrictions place a uniform constraint on the growth of the struts S: $\ell'_{ij} > 1$. Since the system (1) is homogeneous, the system (5–6) is feasible for any choice of right-hand sides in (5). This particular right-hand side has been chosen for convenience in the proof.

The objective function (4) includes the norm of $\mathbf{v}$ as a quadratic term, plus a barrier-type penalty term that keeps the solution away from the the boundary (5) of the feasible region. This penalty term is necessary to achieve a smooth dependence of the solution on the data. Now, because the objective function is strictly convex, and it goes to infinity if $\mathbf{v}$ increases to infinity or approaches the boundary, there is a unique solution $\mathbf{v} =: f(\mathbf{p})$ for every $\mathbf{p}$.

The function $f(\mathbf{p})$ is defined on an open set $U \subset \mathbf{R}^{2n}$ that is characterized by the conditions of Theorem 2: no angles are 0° or 180°, no vertex touches a bar, and at least one cycle is nonconvex or at least one open arc is not straight.

Lemma 6 *f is differentiable on U.*

The proof of this lemma (omitted in this abstract) is based on the stability theory of convex programming under equality constraints.

Proof (Theorem 1): Differentiability of f on U is sufficient to ensure that the initial-value problem

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)), \quad \mathbf{p}(0) = \mathbf{p}_0 \quad (8)$$

has a (unique) *maximal solution* $\mathbf{p}(t)$, $0 \le t < T$, that cannot be extended beyond some positive bound $T \le \infty$; see for example [33, Section II.XXI]. This means that one of three cases occurs:

(a) $\mathbf{p}(t)$ exists for all t, i.e., $T = \infty$.

(b) T is finite, and $\mathbf{p}(t)$ becomes unbounded as $t \to T$.

(c) T is finite, and $\mathbf{p}(t)$ approaches the boundary of U as $t \to T$.

The last case (c) is the case we want: at the boundary of U, some angle becomes straight, and we can reduce the linkage.

Case (a) can be excluded very easily. By assumption, the bar-and-strut framework $G_A(\mathbf{p})$ has some strut $\{i,j\}$ between two points in the same component of the bar framework; their distance increases at least with rate 1, by (5), but it is bounded from above because i and j are linked by a sequence of bars. It follows that the solution cannot exist indefinitely and T must be finite.

If there is a line L that separates the components of the arc-and-cycle set A, this partitions A into two nonempty sets, and each of these can be treated separately and recursively. Unfortunately, the guarantee for the expansive property between different members of the partition is lost. But for the purposes of proving Theorem 1 we may assume that there is no such separation. Then the sum of the maximum diameters of each of the components of A is a uniform a-priori bound on the diameter of A for all time. This eliminates case (b).

Thus we are left with case (c) only. Observe that all pairwise distances of vertices $\mathbf{p}(t)$ are monotonically increasing, and by condition (7) $\mathbf{p}_1$ and $\mathbf{p}_2$ are fixed during the motion. Thus, all other vertices are determined up to reflection, and the whole configuration is determined up to reflection. Thus $\mathbf{p}(t) \to \mathbf{p}$ for some configuration $\mathbf{p}$ as $t \to T$. The configuration $\mathbf{p}$ is on the boundary of U and thus must have some vertex with a straight angle. Then we inductively continue with a simpler linkage. This completes the proof of Theorem 1. □

4.1. Additional Observations

It is useful to notice that, under the appropriate conditions, we can extract more information from the deformation that we have defined in Theorem 1. One observation is that we can preserve any symmetries that the original configuration might have. Consider a group H of congruences of the plane, where each congruence in H fixes a point in the plane. We say that the arc-and-cycle set A has *symmetry group H* if the action of each element of H permutes the vertices and edges of A.

Corollary 1 *If an arc-and-cycle set has a symmetry group H, then there is a piecewise-differentiable proper motion to an outer-convex configuration, such that it is expansive until it is separated and the symmetry group H is preserved during the deformation.*

4.2. Alternative Approaches

There are many ways to select a local motion $\mathbf{v}$ among the feasible local motions whose existence is guaranteed by Theorem 2. We have chosen one possibility that is most convenient for the proof.

As a possible alternative approach, we might consider a *linear* programming problem, with some arbitrary artificial linear objective function $\mathbf{c}$, and some linear normalization condition to ensure boundedness, pinning down some bar $(i_1, i_2) \in B$:

$$\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{i\in V} \mathbf{c}_i \cdot \mathbf{v}_i \\
\text{subject to} \quad & (\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{p}_j - \mathbf{p}_i) = 0 \\
& \qquad \text{for } \{i,j\} \in B, && (9)\\
& (\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{p}_j - \mathbf{p}_i) \ge 0 \\
& \qquad \text{for } \{i,j\} \in S, && (10)\\
& \textstyle\sum_{i\in V} \mathbf{d}_i \cdot \mathbf{v}_i = 1, && (11)\\
& \mathbf{v}_{i_1} = \mathbf{v}_{i_2} = 0, && (12)
\end{aligned}$$

We have given up *strict* expansiveness in (10), The set of vectors given by (9), (10), and (12) forms a polyhedral cone C. Theorem 2 guarantees that there are nonzero solutions. One can check that the normalization conditions (11) ensure that the cone is pointed. The idea is now to use an extreme ray of the cone C for the motion. A vector $\mathbf{d}$ can be found which ensures that the feasible set (9–12) is a bounded set. Any basic feasible solution of the linear program will correspond to an extreme ray of the cone C. It will have a few inequalities of (10) fulfilled with equality. The resulting framework obtained by inserting "artificial" bars corresponding to the nonbasic inequalities of (10), will have a unique vector of velocities $\mathbf{v}$ subject to the normalization constraint (11). This means that the framework is a *mechanism*, allowing one degree of freedom; as the mechanism follows this forced motion, all nonfixed distances will increase, at least for some time.

So one follows the paradigm of parametric linear programming: The optimal basic feasible solution will continue to remain feasible as the coefficients $\mathbf{p}_i$ in the constraints (10) change smoothly. At some point, one of these constraints will threaten to become violated: this is the time to make a *pivot*, exchanging one of the artificial bars for a new one which allows the motion to be continued.

The above discussion has ignored several issues, such as possible degeneracy of the linear program. However, this approach might be more attractive from a conceptual, as well as a practical point of view.

Recently, Streinu [31] has found a class of such mechanisms, so-called *pseudo-triangulations*. These structures have some nice properties; for example, they form a planar framework of bars. Streinu [31] claims that a polygonal arc can be opened by a sequence of at most $O(n^2)$ motions, where each motion is given by the mechanism of a single pseudo-triangulation.

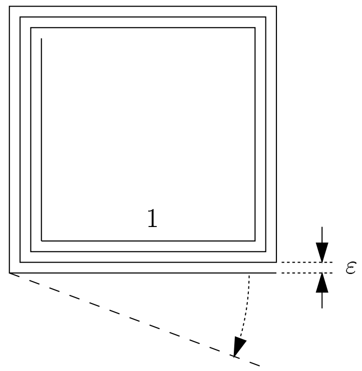

Figure 8. An arc that is numerically difficult to unfold.

4.3. Comparison of Approaches

The approach based on mechanisms might avoid some of the numerical difficulties associated with solving the optimization problem (4–6). For example, consider a spiral n-bar arc winding around a unit square in layers of thickness ε (Figure 8). In the solution of (5–6), a rough estimate shows that the outermost vertex must move with a speed of at least ε^{4-n}, as $\varepsilon \to 0$. On the other hand, the "natural" solution of unwinding the spiral one bar at a time fits nicely into the setup of mechanisms and the parametric linear program approach.

Our proof has certain nonconstructive aspects: the direction $\mathbf{v}$ of movement is specified implicitly as the solution of an optimization problem, and the global motion arises as the solution of a differential equation. Both of these items are numerically well-understood, and our approach lends itself to a practical implementation. Indeed, we implemented our approach to produce animations such as Figure 1. However, this does not necessarily lead to a finite algorithm in the strict sense. The optimization problem (4–6), having an objective function which is rational, can in principle be solved exactly by solving a system $h(p, x, \lambda) = 0$ of algebraic equations. The differential equation cannot be solved explicitly, but it may be possible to bound the convergence and develop a finite algorithm for a digital computer (solving the differential equation up to a given error bound).

Since the motions of a mechanism are described by algebraic equations, Streinu's algorithm leads to a finite algorithm for a digital computer, at least in principle. It remains to be seen how a practical implementation competes with our approach; in any case, as an algorithm for a direct realization of the motion by a mechanical device, Streinu's algorithm appears attractive.

On the other hand, the nonlinear programming approach might be preferable because it produces a "canonical" movement. In particular, when the starting configuration is symmetric, this symmetry will be maintained throughout the whole motion (Corollary 1).

5. Related Problems

In this section we settle a few natural questions related to our main theorem.

Theorem 6 *Any smooth expansive noncongruent motion of a simple closed polygonal curve C in the plane, fixing the lengths of its edges, must increase the area of the interior of C during the motion.*

5.1. Topology of Configuration Spaces

It is natural to ask more about the structure of the *configuration space* of an arc-and-cycle set. Let $X(G, L)$ denote the space of all configurations of embeddings in the plane of of a bar graph G consisting of a finite number arcs and cycles, without self-intersections, where the edge lengths are determined by $L = (\dots, \ell_{i,j}, \dots)$. This inherits a natural topology from considering all the coordinates of all the vertices as part of a large dimensional Euclidean space. Let $X_0(G, L) \subset X(G, L)$ denote the subspace of outer-convex configurations. We assume that L is chosen so that there is at least one realization in the plane.

Theorem 7 *The space of outer-convex realizations $X_0(G)$ is a strong deformation retract of $X(G)$.*

The main point to remember is that the limit in Theorem 1 depends continuously on the initial starting configuration. The following is a natural consequence of Theorem 7.

Corollary 2 *If the underlying graph G is a single arc or a single cycle, then $X(G, L)$ modulo congruences (including orientation reversing ones) is contractible.*

Here the main task is to show that the space of convex realizations is contractible.

5.2. Open Problems

Another direction is to explore what happens when the arc-and-cycle set is allowed to touch but not cross:

Conjecture 1 *If G is a single arc or a single cycle, then the closure of $X(G, L)$ modulo congruences is contractible.*

We conjecture that motions can be realized by a sequence of relatively simple motions:

Conjecture 2 *If A is an arc-and-cycle set in the plane, then there is a flex that takes it to an outer-convex configuration, by a finite sequence of motions, where each motion changes at most four vertex angles.*

It also remains open precisely how many such moves are needed.

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One of the key meetings that started this paper is the Monte Verità Conference on Discrete and Computational Geometry in Ascona, Switzerland, organized by Jacob Goodman, Richard Pollack, and Emo Welzl in June 1999. In particular, the authors first met at this conference. Our work continued at the 4th Geometry Festival, an international workshop on Discrete Geometry and Rigidity, in Budapest, Hungary, organized by András Bezdek, Károly Bezdek, Károly Böröczky, and the first author, in November 1999. We thank the organizers of these workshops, in particular for the opportunity to meet and work together.

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