A Disk-Packing Algorithm for an Origami Magic Trick *

M. Bern
Xerox Palo Alto Research Center, USA

E. Demaine
University of Waterloo, Canada

D. Eppstein
University of California - Irvine, USA

B. Hayes
Placeware Inc., USA

Abstract

We present an algorithm for a magic trick. Given a polygon with holes P, our algorithm determines a folding of a rectangular sheet of paper such that a single straight cut suffices to cut out P.

Keywords: Computational Geometry, Paperfolding, Polygon.

1 Introduction

The great Harry Houdini was one of the first to perform the following magic trick: fold a sheet of paper so that a single straight cut produces a cut-out of a rabbit, a dog, or whatever else one likes. Whereas Houdini only published a method for a five-pointed star [11] (a method probably known to Betsy Ross [10]), Martin Gardner [9] posed the question of cutting out more complex shapes. Demaine and Demaine [5] stated this question more formally: given a polygon with holes P (possibly with more than one connected component) and a rectangle R large enough to contain P, find a "flat folding" of R such that the cross-section of the folding with a perpendicular plane is the boundary of P. More intuitively, a single straight cut of the flat folding produces something that unfolds to P. A flat folding [4, 12]

*David Eppstein supported in part by NSF grant CCR-9258355 and matching funds from Xerox Corp. Erik Demaine supported in part by NSERC.

is a mathematical notion, abstracting folded paper to a nonstretchable, non-self-penetrating, zero-thickness, piecewise-linear surface in $\mathbb{R}^3$.

Demaine et al. [6] have proposed a solution to this *cut-out problem*, based on propagating paths of folds out to the boundary of the rectangle R. Here we give a more "local" solution, based on disk packing. Our strategy is to pack disks on R so that disk centers induce a mixed triangulation/quadrangulation respecting the boundary of polygon P. We fold each triangle or quadrilateral interior (exterior) to P upwards (respectively, downwards) from the plane of the paper, taking care that neighboring polygons agree on crease orientations. A cut through the plane of the paper now separates interior from exterior.

Disk packing has previously been used to compute triangulations [1] and quadrangulations [3] with special properties. Disk packing, or more precisely disk placement, has also been applied to origami design, most notably by Lang [13]. In fact, the result in this paper is in some sense a fusion of a quadrangulation algorithm from Bern and Eppstein [3] with an origami design algorithm from Lang [13].

2 Disk Packing

Let P be a polygon with holes, strictly contained in a rectangle R. We think of P as boundary along with interior. Let PR denote the planar straight line graph that is the union of the boundary of P and the boundary of R. In this section, we show how to pack disks such that each edge of PR is a union of radii of disks, and such that the disks induce a partition of R into triangles and quadrilaterals. Our solution is closely related to some mesh generation algorithms [1, 3].

The disk packing starts with interior-disjoint disks. We call a connected portion of R minus the disks a *gap*. We call a gap bounded by three arcs a *3-gap* and one bounded by four arcs a *4-gap*. We begin by centering a disk at each vertex, including the corners of R. At vertex v, we place a disk of radius one-half the distance from v to the nearest edge of PR not incident to v. We introduce a subdivision vertex (a degree-2 vertex with a straight angle) at each intersection of a disk boundary and an edge of PR.

Now consider the edges of (the modified) PR that are not covered by disks. Call such an edge *crowded* if its diameter disk intersects the diameter disk of another edge of PR. We mark each crowded edge, and then split each crowded edge by adding its midpoint. We continue marking and splitting in any order until no edges of PR are crowded. We then add the diameter disk of each PR edge so that each edge is a union of diameters of disks as required. Strictly speaking, only the edges of P need be covered by disks, but we include the boundary of R mainly for the sake of "neatness".

Next we add disks until all gaps between disks are either 3-gaps or 4-gaps. This can be done by computing the Voronoi diagram of the disks placed so far, and repeatedly placing a maximal-radius disk at a Voronoi vertex and then updating the

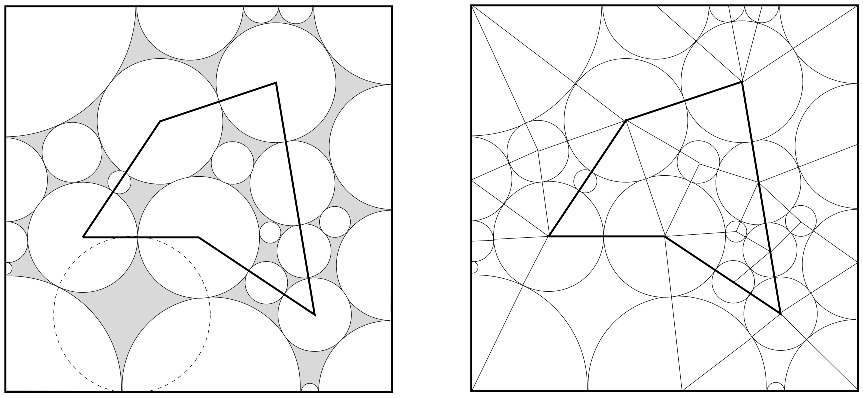

Figure 1: (a) A disk packing respecting the boundary of the polygon. Vertices of 4-gaps are cocircular. (b) Induced triangles and quadrilaterals.

Voronoi diagram. Bern et al. [1] give an $O(n \log^2 n)$ algorithm for this process; Eppstein [8] improves this time to $O(n \log n)$. For our purposes, n denotes the number of disks used to cover the edges of PR.

Figure 1(a) gives an example disk packing, not precisely the same as the one that would be computed by the algorithm just given. By adding edges between the centers of tangent disks, the disk packing induces a decomposition of R into triangles and quadrilaterals as shown in Figure 1(b).

3 Molecules

A *molecule* is a (typically flat) folding of a polygon that can be used as a building block in larger origamis. We shall fold the triangles in the decomposition of R with *rabbit ear molecules*. In the rabbit ear molecule, a mountain fold meets each of the triangle's vertices; these folds lie along the angle bisectors of the triangle so that the boundary of the triangle is coplanar in the folded "starfish". A valley fold meets each of the triangle's sides at the points of tangency of the disks; these all fold to a vertical *spine*, perpendicular to the original plane of the paper. At this point, we regard the orientations of the folds as changeable: in the larger origami some of them may be flat (not folded) or reversed from their initial assignment. For example, to form a flat origami from a single rabbit ear, one could reverse one of the valleys into a mountain in order to satisfy Maekawa's theorem.[1] The boundary of the triangle becomes collinear in the flat origami.

[1] Maekawa's theorem for flat origami [4, 12] states that at any vertex interior to the paper the number of mountains minus the number of valleys must be plus or minus two.

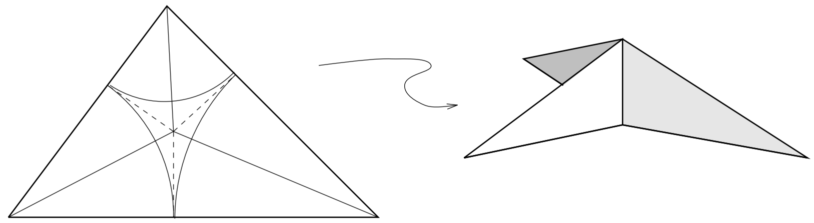

Figure 2: A rabbit ear molecule folds into a three-armed “starfish”.

We shall fold the quadrilaterals as shown in Figure 3. In this molecule, a relative of the standard *Husimi molecule* [13], mountain folds lie along the angle bisectors and valley folds lie along the line segments from the meeting point of the angle bisectors to the points of tangency of the disks. The folds so far described produce a four-armed starfish, but not one that can be turned into a flat origami with collinear boundary. In the flat folding produced by, say, reversing one of the valley folds, opposite arms fold into collinear pairs. To fix this difficulty, we “crimp fold” the two lower arms so that the bottom edges of all four arms are collinear in the flat folding. We note in passing that the standard Husimi molecule, which places the valley folds along perpendiculars to the sides, gives a collinear boundary without any additional folds; however, we cannot use this standard molecule, because then neighboring molecules would not match up.

In our folding of the quadrilateral, we are using a special property of quadrilaterals induced by 4-gaps: the four vertices of a 4-gap, that is, the points of tangency of the four disks, are always cocircular [1]. Figure 1(a) shows the circle for one 4-gap. This property guarantees that the angle bisectors of the quadrilateral all meet at a point, namely the center of the circle through the tangencies. It also guarantees that the four valley folds all have the same the length, so they meet to form the spine of the folded starfish. Not all properties of triangles, however, carry over to quadrilaterals induced by 4-gaps: the spine of the four-armed starfish is not perpendicular to the collinear boundary in the flat folding.

4 Joining Molecules

We now show how to assign final orientations to creases, so that neighboring molecules fit together and each vertex satisfies Maekawa’s theorem. This fills in (a special case of) a missing step in Lang’s algorithm [13].

We are aiming for a final folding of R that resembles a book of flaps, something like the rightmost picture in Figure 3. More precisely, the folding will look like two books of triangular flaps, one above and one below the original plane of the paper. The molecules (triangles and quadrilaterals) inside P will form the top book, whereas those outside P will form the bottom book. The boundary of P it-

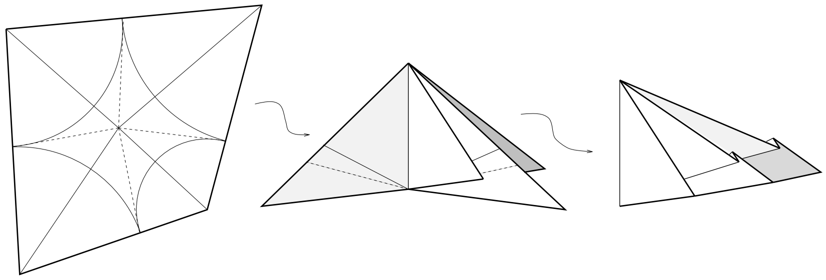

Figure 3: We fold a quadrilateral into a starfish with two pairs of parallel arms, and then crimp the lower arms to meet the upper arms.

self will not be folded, and the polygons crossing the boundary, each containing a triangle from two different original molecules, will thus belong to both books.

All *angle bisector edges* inside P will be mountains, whereas those outside P will be valleys. Other edges of the crease pattern receive *default orientations*, subject to reversal in a final matching step. The default orientation of a *tangency edge* (an edge to a point of tangency) or a *side edge* (an edge along the side of a triangle or quadrilateral) is valley inside P and mountain outside P. Side edges lying along the boundary of P are not folded at all.

At this point, each vertex of the crease pattern has an equal number of mountains and valleys. The vertices interior to R inside P need one more mountain, whereas those outside P need one more valley, in order that molecules fold to their assigned half-spaces, above or below the original plane of the paper. (Vertices on the boundary of P can have an excess of either mountains or valleys.) Let G be the planar graph obtained from the decomposition by removing all angle bisector edges and all edges along the boundary of P. We would like to find a set of edges M—a matching—such that each vertex of G lying in the interior of R is incident to exactly one edge of M. By reversing the orientations of the edges of M, we ensure that each vertex satisfies Maekawa's theorem. All vertices, even the ones along P, which lost two edges each from the original decomposition of R, also satisfy Kawasaki's theorem.[2]

We now show how to solve the matching problem using dual spanning trees. Let T_C be a tree of side edges such that: T_C includes no edges along the boundary of R or P; T_C spans all interior corners of molecules; and T_C spans exactly one corner along the boundary of R, which we consider to be its root. If we were to cut the paper along T_C, we would obtain a tree of molecules T_M, as shown in Figure 4(a). We root T_M at one of the molecules incident to the root of T_C. The matching M contains two types of edges: each tangency edge from the center of a

[2] Kawasaki's theorem for flat origami [4, 12] states that at any vertex interior to the paper the sum of alternate angles must be $180°$.

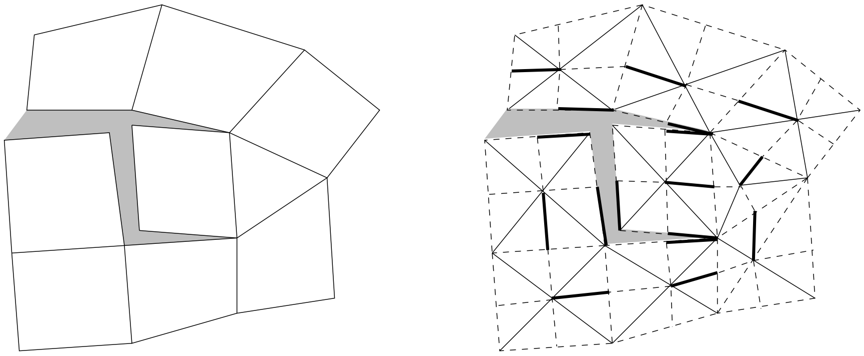

Figure 4: (a) Cutting out a tree T_C (shaded) spanning interior corners leaves a tree of molecules T_M. Roots are at the upper left. (b) The matching consists of side edges from corners to parents in T_C and tangency edges from molecule centers to parents in T_M. Assignments shown assume all molecules are inside P.

molecule to the side of its parent in T_M (along with one such edge inside the root molecule), and each side edge from a corner to its parent (a tangency point) in tree T_C. See Figure 4(b).

To picture the effect of this choice of M on the eventual flat folding, imagine that we have actually cut along the edges of T_C. Imagine building up the flat folding molecule by molecule in a preorder traversal of T_M. The root molecule of T_M folds to a book of flaps with collinear edges lying along the original plane of the paper. Each child molecule adds a "pamphlet" of three or four flaps between two flaps of the book we have constructed so far. The cover and back cover of the pamphlet are glued to their adjacent pages, so that a quadrilateral thickens two old flaps and adds two new flaps. Since in general the spines of the book and the pamphlet are not perpendicular to the plane of the paper, we must sometimes make some additional folds in order to fit the pamphlet inside the book. We defer the solution to this problem until later; for now let us pretend that all spines are indeed perpendicular.

We continue gluing pamphlets between flaps of the growing book as we go down the tree. Whenever we cross the boundary of P, we glue the next pamphlet above or below—rather than between flaps of—its parent molecule, so that the boundary of P is not itself folded. When we are done joining all the molecules we indeed have two books of flaps, one above and one below the original plane of the paper.

Now imagine taping the cut edges back together in a postorder traversal of T_C. Before taping, the cut leading to a leaf of T_C, say inside P, defines the bottom edge of two adjacent "armpits", as shown in Figure 5. (An armpit consists of one layer from each of two adjacent flaps.) Taping together the first and last layer of

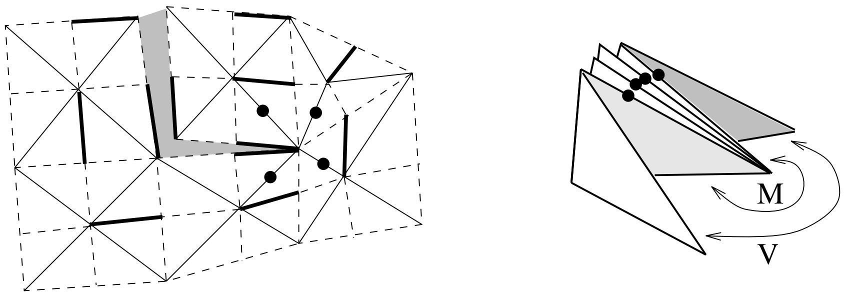

Figure 5: Taping together a cut leading to a leaf of T_C amounts to joining two "armpits" in the book of flaps.

the intervening flap forms a mountain fold, agreeing with the orientation we gave to side edges in the matching. Taping together the remaining two sides of the cut forms a valley fold, agreeing with the default orientation of side edges inside P. Taping a cut leading to a leaf of T_C closes two armpits and reduces the number of flaps in the book by two. We can continue taping cuts all the way up T_C. Since each taping joins armpits adjacent at the time of the taping, there can be no "crossed" pair of tapings, or put another way, no place where the paper is forced to penetrate itself.

We now return to the remaining problem: incompatible spines. The crease pattern just described, even though it satisfies Maekawa's and Kawasaki's theorems at each vertex, may not yet fold flat. The problem is that a molecule with a steep (close to vertical) spine cannot fit beneath the "overhang" of a book with a shallow spine without the paper penetrating itself. To solve this problem, we add a fan of pleats to a steep spine as shown in Figure 6, so that it fits beneath every flap overhanging it. (Once we have defined an overhang order with the tree T_M, we can if we like pre-fold all molecules to have spines of the same slope.) With this final modification to the crease pattern, the paper R can indeed be folded flat so that P lies above, and its complement $R \setminus P$ lies below, the original plane of the paper.

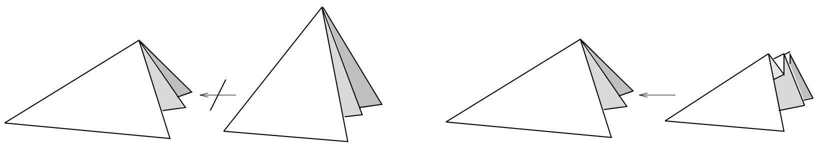

Figure 6: We fit a molecule beneath an overhang by fan pleating its spine.

5 Fattening the Polygon

At this point, we have a degenerate solution to the cut-out problem. A cut through the original plane of the paper separates P from its complement. Unfortunately, it also cuts P into its constituent molecules. A cut very slightly below the original plane of the paper leaves P intact, while adding a small "rim" to P.

We can remove the degeneracy by fattening the boundary of P into a narrow "ribbon" as shown in Figure 7. The boundaries of the ribbon are slightly inside and outside the original P; vertices of P are moved in or out along angle bisectors. The width of the ribbon must be smaller than the *minimum feature size* of the polygon, the minimum distance between a vertex of P and an edge not incident to that vertex.

We modify the disk packing step so that it packs partial disks (sectors) around the boundary of ribbon, such that interior and exterior sectors match up. Creases between corresponding subdivision points cross the ribbon at right angles, whereas creases between corresponding vertices cross at angle bisectors, so that each vertex still satisfies Kawasaki's theorem. Notice that interior and exterior sectors centered on corresponding vertices have slightly different radii.

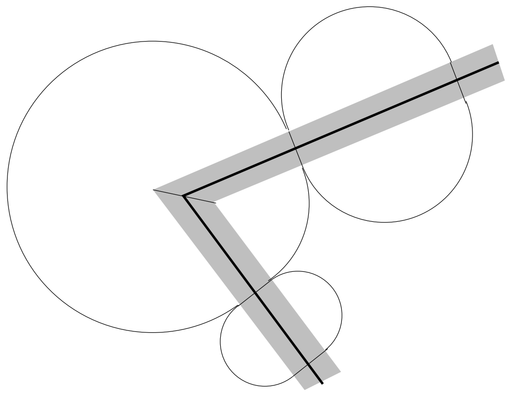

Figure 7: Fattening the polygon into a ribbon lets P survive the cut intact.

6 Discussion

We have given an algorithm for the cut-out problem. More precisely, we have given an algorithm for computing a crease pattern with a flat folding that solves the cut-out problem. We have not described how to actually transform the crease pattern into the flat folding.

Is our algorithm usable? The answer is a qualified yes. Figure 8 gives a crease pattern for a fish cut-out that is not too hard to fold. In this crease pattern, we have taken a number of shortcuts to make the algorithm more practical. First, we have used only three-sided and special four-sided gaps, ones in which the valley folds from the center vertex meet the sides perpendicularly. Second, we have not packed the disks all the way to the boundary of the paper, only far enough that radiating folds do not meet within the page. Third, we have not fattened the polygon, and hence the cut should be placed slightly below original plane of the paper, so that the interior remains connected.

The number of disks used by our algorithm (and hence, up to constant factors, the number of molecules) is not really excessive, linear in the size of the subdivision of P. The size of the subdivision, in turn, is a fairly natural complexity measure of the polygon. Define the *local feature size* $LFS(p)$ at a point p on an edge e of P to be the distance to the closest edge that is not adjacent to e [14]. Then it is not hard to show that the number of disks is $O(\int 1/|LFS(p)|)$, where the integral is over the boundary of P.

The fan pleating may add significantly more creases, depending upon the slopes of the spines, which in turn depend on the shapes of 3- and 4-gaps. The fan pleating folds a starfish's excess paper between the desired spine and the eventual cutting plane. If the polygon P is fattened into a ribbon of width proportional to the minimum feature size, then the spacing between fan pleats is also of the same order as the minimum feature size.

The algorithm of this paper can be generalized to the problem in which the input is a planar straight-line graph G, and a single cut must cut along all the edges of G. An interesting open question asks whether there is a polynomial-size solution (polynomial in the number of original vertices of P or G) for the cut-out problem. A solution using disk packing may shed some light on two other computational geometry problems: simultaneous inside-outside nonobtuse triangulation [2] and conforming Delaunay triangulation [7].

Marshall Bern is a member of the research staff at Xerox Palo Alto Research Center, 3333 Coyote Hill Rd., Palo Alto, CA 94304, USA. E-mail: bern@parc.xerox.com

Erik Demaine is a graduate student in the Department of Computer Science at the University of Waterloo, Waterloo, ON N2L 3G1, Canada. E-mail: eddemaine@uwaterloo.ca

David Eppstein is a professor in the Department of Information and Computer Science at the University of California, Irvine, CA 92697, USA. E-mail: eppstein@ics.uci.edu

Barry Hayes is a senior engineer at Placeware, Inc., 201 Ravendale Dr., Mountain View, CA 94043, USA. E-mail: bhayes@cs.stanford.edu

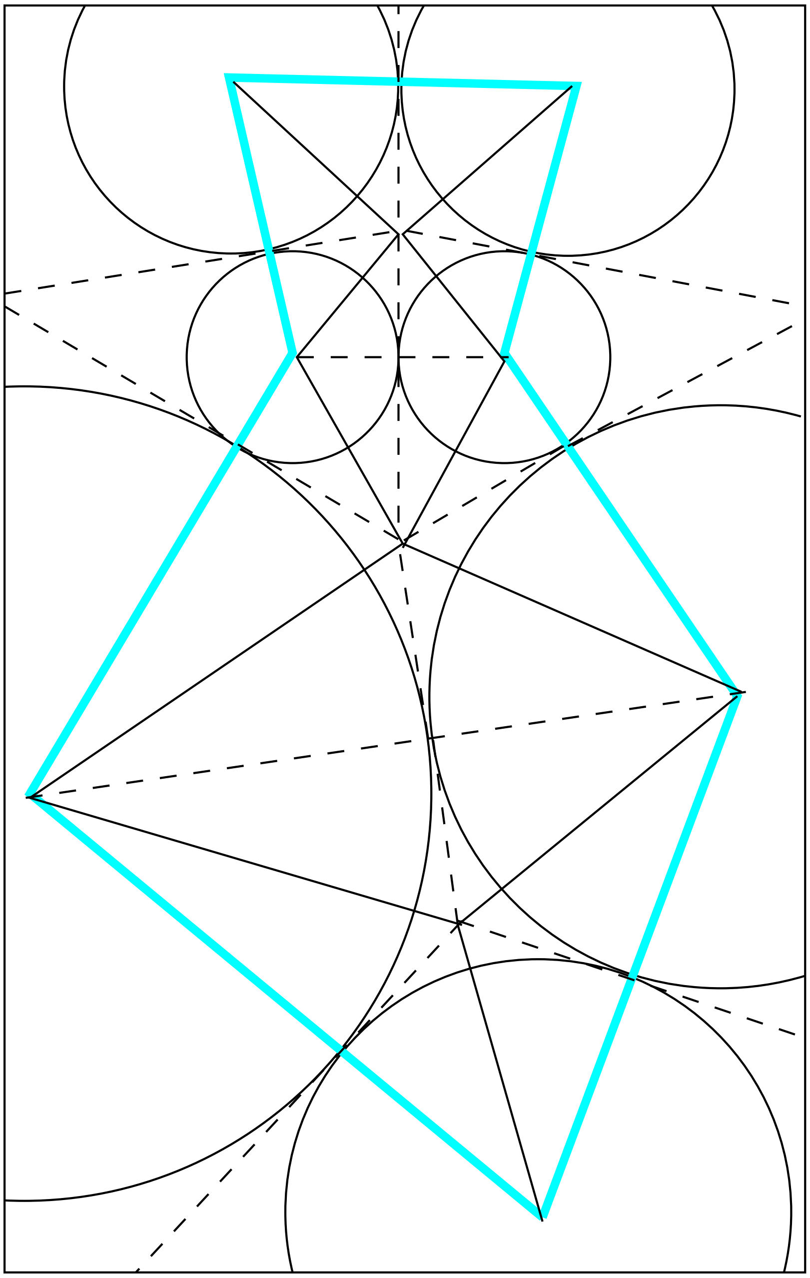

Figure 8: An example for the reader to try. This design incorporates some practical shortcuts. For example, packing the exterior with disks is unnecessary, because radiating folds do not collide.

References

[1] M. Bern, S. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulation of polygons. *Disc. Comput. Geom.* 14 (1995) 411–428.

[2] M. Bern and D. Eppstein. Polynomial-size nonobtuse triangulation of polygons. *Int. J. of Comp. Geom. and Applications* 2 (1992) 241–255.

[3] M. Bern and D. Eppstein. Quadrilateral meshing by circle packing. *Proc. 6th Int. Meshing Roundtable*, 1997, 7–19.

[4] M. Bern and B. Hayes. The complexity of flat origami. *Proc. 7th ACM-SIAM Symp. Disc. Algorithms*, 1996, 175–183.

[5] E. Demaine and M. Demaine. Computing extreme origami bases. Tech. Report CS-97-22, Dept. of Computer Science, U. of Waterloo, May 1997.

[6] E. Demaine, M. Demaine, and A. Lubiw. Folding and one straight cut suffice. Tech. Report CS-98-18, Dept. of Computer Science, U. of Waterloo, July 1998.

[7] H. Edelsbrunner and T.-S. Tan. An upper bound for conforming Delaunay triangulations. *Disc. Comput. Geom.* 10 (1993) 197–213.

[8] D. Eppstein. Faster circle packing with application to nonobtuse triangulation. *Int. J. Comp. Geom. & Applications* 7 (1997) 485–491.

[9] M. Gardner. Paper Cutting. In *New Mathematical Diversions*, Math. Assoc. America, 1995, 58–69.

[10] National Standards and Emblems. *Harper's New Monthly Magazine* 47 (1873) 171–181.

[11] H. Houdini. In *Paper Magic*, E. P. Dutton & Company, 1922, 176–177.

[12] T. Hull. On the mathematics of flat origamis. *Congressus Numerantium* 100 (1994) 215–224.

[13] R. Lang. A computational algorithm for origami design. *Proc. 12th ACM Symp. Comp. Geometry*, 1996, 98–105.

[14] J. Ruppert. A new and simple algorithm for quality 2-dimensional mesh generation. *Proc. 4th ACM-SIAM Symp. Disc. Algorithms*, 1993, 83–92.