Polygons Flip Finitely: Flaws and a Fix

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Abstract

Every simple planar polygon can undergo only a finite number of pocket flips before becoming convex. Since Erdős posed this as an open problem in 1935, several independent purported proofs have been published. However, we uncover a plethora of errors and gaps in these arguments, and remedy these problems with a new (correct) proof.

1 Pocket Flips

Given a simple polygon in the plane, a *pocket* is a maximal connected region interior to the convex hull and exterior to the polygon. A *(pocket) flip* is the reflection of a pocket, or more precisely the subchain of the polygon bounding the pocket, across its line of support, the bounding edge of the convex hull. In 1935, Paul Erdős [3] introduced the problem of simultaneously flipping all pockets of a simple polygon, and repeating this process until the polygon becomes convex. He conjectured that this process terminates after a finite number of steps. In 1939, Béla Nagy [2] pointed out that flipping multiple pockets simultaneously may make the polygon nonsimple. Modifying the problem slightly, he argued that repeatedly flipping one pocket of the current polygon always convexifies the polygon after a finite number of flips.

This result has come to be known as the Erdős-Nagy Theorem. Over the years, the theorem has been rediscovered many times, each discovery leading to a new proposed proof. Among the arguments published in English, some are long and technical, others use higher mathematics, some prove a weaker result, some leave gaps for the reader to fill, and still others are incorrect. In Section 2, we describe these arguments and point out their weaknesses, gaps, and errors. We view only one proof, by Kazarinoff and Bing [8, 1, 7], as completely correct, though terse. Then, in Section 3, we present our own proof which attempts to combine the most elegant portions of the existing arguments, along with a few new tricks, into a (correct) proof that we believe is both simple and thorough.

2 Existing Arguments

We begin by introducing some notation used in this paper. Let $d(x,y)$ denote the Euclidean distance between points x and y. Call a vertex of a simple polygon *flat* if its interior angle is π. Let $P = P^0 = \langle v_0, v_1, \ldots, v_{n-1}\rangle$ denote the initial polygon and its vertices. Let $P^k = \langle v_0^k, v_1^k, \ldots, v_{n-1}^k\rangle$ denote the resulting "descendant" polygon after k arbitrary pocket flips; if P^k is convex for some k, then we define $P^k = P^{k+1} = P^{k+2} = \cdots$. Let C^k denote the convex hull of P^k. When we talk about convergence, it is always with respect to $k \to \infty$. When the limit of P^k exists, we denote it by P^*, its vertices by v_i^*, etc.

2.1 Nagy

The very first claimed proof, published by Béla de Sz.-Nagy in 1939 [2], is brilliant in overall design, but unfortunately has a fatal flaw that may have gone undetected until now.[1] Nagy's argument consists of the following main steps:

1. The sequence P^k converges.
2. The limit P^* is convex.
3. Nonflat vertices of P^* converge in finite time.
4. The sequence P^k converges in finite time.

The flaw is in Step 2, where Nagy uses the claim that $P^0 \subseteq C^0 \subseteq P^1 \subseteq C^1 \subseteq \ldots$ to show that P^k and C^k converge to the same (necessarily convex) limit. As illustrated in Figure 1, this claim is incorrect. When there are multiple pockets to choose from, $C^k \not\subseteq P^{k+1}$.

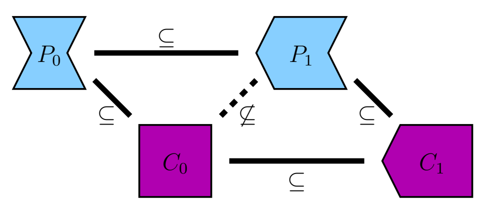

Figure 1: Nagy's error: $P^0 \subseteq C^0 \not\subseteq P^1 \subseteq C^1$.

Despite most later arguments being based on Nagy's, this flaw seems unique to Nagy's argument. Many later arguments use the other steps of Nagy's argument, to which we now turn.

In Step 1, Nagy observes that the perimeter of P^k is constant, and concludes that each v_i^k has a point of accumulation. Then he observes that, for x inside P^k and $m \geq k$, $d(x, v_i^m) < d(x, v_i^{m+1})$. Therefore, for $n \geq m$,

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[1] We should point out, though, that Grünbaum [4] states that Bing and Kazarinoff [1] remark that Nagy's proof [2] is invalid. However, Grünbaum [4] goes on to say that there is no basis for this claim. We have not yet seen the Russian paper [1] and thus cannot assess this point further.

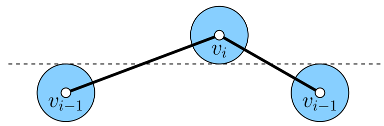

Figure 2: For a nonflat vertex v_i^k, once all the vertices are within a small enough ball around their limit, there is a line which separates the ball of v_i^k from all the other balls. Thus v_i^k subsequently remains on the convex hull of P^k and cannot be flipped again.

$d(v_i^m, v_i^n) < d(v_i^m, v_i^{n+1})$, which prevents the existence of multiple points of accumulation, thus proving convergence.

To prove Step 3, Nagy uses an argument illustrated in Figure 2 to show that nonflat vertices of the limit polygon converge in finite time. This argument is easy to draw, but requires care to justify in detail, while Nagy's presentation is somewhat terse.

Finally, in Step 4, once all the nonflat vertices have converged, no more flips are possible, because they would cause the convex hull to increase beyond its limit.

2.2 Grünbaum

Branko Grünbaum [4] described some of the intricate history of this problem following the appearance of Nagy's paper [2], uncovering several rediscoveries of the theorem. He also provided his own argument, similar to Nagy's but more terse. One main difference is that, at each step, he flips the pocket that has maximum area (if there is more than one pocket to choose from). Therefore Grünbaum [4] actually proves a weaker theorem: there exists a (well-chosen) sequence of flips that convexifies after finitely many flips. An extended version of [4] was published in 2001 by Grünbaum and Zaks [5].

Grünbaum's argument has a similar structure to Nagy's:

1. A subsequence of the sequence P^k converges to a convex limit.
2. The whole sequence converges.
3. Nonflat vertices of the limit polygon converge in finite time. (Same proof as Nagy.)
4. The sequence converges in finite time. (Same as Nagy.)

For Step 1, Grünbaum invokes Nagy's "constant polygon length" argument to show that a subsequence converges. He then claims that "due to the selection of pockets that maximize the area, the limit polygon P^* is convex," without further explanation. We view this unjustified claim as a gap in the proof, because the convexity of P^* has been a stumbling block in most claimed proofs of the theorem.

In Step 2, Grünbaum invokes Nagy's "distances from points in the polygon increase" observation, without further justification. As in Nagy's proof, this argument seems insufficient by itself, requiring more detail.

2.3 Reshetnyak and Yusupov

In 1957, two papers in Russian by Reshetnyak [9] and Yusupov [13] claimed proofs of the theorem. According to Grünbaum [4], these arguments are similar to Nagy's [2]. We have not yet studied the differences in detail.

2.4 Kazarinoff and Bing

In 1959, Kazarinoff and Bing [8] announced the problem and a solution. Two years later, a proof appeared in a paper by Bing and Kazarinoff [1] and also in Kazarinoff's book [7]. They also conjectured that every simple polygon becomes convex after at most $2n$ flips. This conjecture has since been shown to be false; see Section 2.5.

The proof described in Kazarinoff's book [7] has no missing steps, and suffers only from being terse. Our proof distinguishes itself mainly by providing more detail. Their proof proceeds as follows:

1. The sequence P^k converges to a limit P^*.
2. Nonflat vertices of the convex hull of P^* converge in finite time.
3. All vertices of P^k converge in finite time. (Same idea as Nagy.)

For Step 1, Kazarinoff and Bing use the "constant polygon length" and the "distances from points in the polygon increase" arguments. They show that, for x interior to C^0, the sequence $d(x, v_i^k)$ is bounded and monotonic, and thus it converges. Applying this argument for three noncollinear points x_1, x_2, and x_3 shows that each v_i^k converges to the unique intersection of three circles.

In Step 2, they argue that, because P^k converges, its interior angles must also converge. Thus, any vertex that converges to a nonflat vertex of C^* has an interior angle less than π after a finite number of steps. Because a vertex moves only when it is flipped, and a flip changes an interior angle α into the angle $2\pi - \alpha$, the vertex can no longer move.

2.5 Joss and Shannon

In 1973, two students of Grünbaum at the University of Washington, R. R. Joss and R. W. Shannon, worked on this problem but did not publish their results. Grünbaum [4] gives an account of the unfortunate circumstances surrounding this event. They found a counterexample to the conjecture of Bing and Kazarinoff (but unaware of the conjecture). Specifically, they showed that, given any positive integer k, there exist simple polygons of constant size (indeed, quadrilaterals suffice) that cannot be convexified with fewer than k flips. See [4, 11].

2.6 Wegner

In 1981, Kaluza [6], apparently unaware of the previous work, posed the problem again and asked whether the number of flips could be bounded as a function of the number of polygon vertices. In 1993, Bernd Wegner [12] took up Kaluza's challenge and solved both problems again. His proof of convexification in a finite number of flips is quite different from the others, but his example of unboundedness is the same as that of Joss and Shannon.

Wegner's proof is certainly the most intricate of the proofs we have seen. His proof is very technical, for example, using convergence results from the theory of convex bodies, and difficult to summarize. To his credit, Wegner carefully details his reasoning, unlike many other authors.

Wegner's approach contains a number of new ideas. He notices that the undirected angles between consecutive polygon edges are monotonically nondecreasing, as they only change when a vertex is on the edge of the lid being flipped. The use of undirected angles makes this property stand out, but prevents the use of angles to show convergence in finite time as in Kazarinoff's proof [7].

Instead, Wegner introduces the area A^k of P^k and tries to show that, after a finite number of flips, performing an additional flip would cause A^k to exceed the area A^* of its limit. He lower-bounds the increase in area during a flip that moves vertex v_i^k by considering the area a of the triangle $v_{i-1}^k v_i^k v_{i+1}^k$. Wegner argues that, during such a flip, A^k will increase by at least $2a$, and uses this fact to force A^k beyond its limit. However, as illustrated in Figure 3, the increase in area by $2a$ occurs only for reflex vertices v_i^k. Fortunately, this flaw is easy to fix, because a convex vertex becomes reflex after one flip, so the next time it moves, Wegner's argument indeed applies.

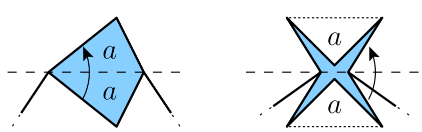

Figure 3: Flipping a reflex vertex increases the polygon area by twice the area a of the incident triangle (left), but this property is not true of a convex vertex (right).

2.7 Toussaint

Motivated by the desire to present a simple, clear, elementary, and pedagogical proof of such a beautiful theorem, Toussaint [10] presented a more detailed and readable argument at CCCG 1999. He combined Kazarinoff and Bing's approach to proving the convergence of P^k with Nagy's approach of proving that convergence occurs in finite time.

The original argument that appeared in [10] uses one instead of three noncollinear points x_1, x_2, and x_3 to conclude that the vertices v_i^k converge. However, without further justification, it is plausible that v_i^k circles around x and thus has multiple accumulation points. Because Toussaint's argument is explicit in the details, this issue is clearly an error. (This is unlike Grünbaum's argument where the reader is left guessing whether Grünbaum was in error or left some trick unreported.) This led the first and third authors of this paper to point out the problem, and propose the three-point solution. This correction appeared in the journal version of Toussaint's argument [11].

Unfortunately, both arguments [10, 11] make an invalid deduction for establishing the convexity of the limit polygon P^*: "we note that the limit polygon must be convex, for otherwise, being a simple polygon, another flip would alter its shape contradicting that it is the limit polygon." For some intuition on why this deduction is invalid, imagine that there are two portions of the polygon that each inflate infinitely often (hypothetically, of course). If we spend all of our time flipping just one of those portions, the other portion never gets flipped, so the limit is nonconvex.

3 Proof of the Erdős-Nagy Theorem

We now offer a short, elementary, and self-contained proof of the Erdős-Nagy Theorem. After writing our proof, we discovered that it uses essentially the same arguments as Kazarinoff and Bing [8]. The main difference is that we endeavored to detail all important steps. As we shall see in Section 3.1, this led to some small changes from [8] which we feel enhance the clarity of the proof.

Theorem 1 *A simple polygon P can undergo only a finite number of pocket flips before being convexified.*

Proof. Reasoning by contradiction, suppose that there were an infinite sequence of polygons $P^k = \langle v_0^k, \ldots, v_{n-1}^k \rangle$, indexed by k, each P^k derived from the previous P^{k-1} by exactly one pocket flip (i.e., the sequence P^k never becomes constant), starting from $P^0 = P$. Let x be any point inside P. By definition of flipping, we have $P^0 \subset P^1 \subset \cdots \subset P^k$, so x is inside all descendants of P.

We first offer an outline of the proof:

1. The distance from each vertex v_i^k to a fixed point $x \in P$ is a monotonically nondecreasing function of k.
2. The sequence P^k approaches a limit polygon P^*.
3. The angle θ_i^k at vertex v_i^k converges.
4. Any vertex v_i^k that moves infinitely many times converges to a flat vertex v_i^*.
5. The infinite sequence P^k cannot exist.

Step 1. First we prove that the distance from x to any particular vertex v_i is monotonically nondecreasing with k. Let $d(x, v_i^k)$ be this distance at step k. If the $(k+1)$st flip does not move v_i, then the distance remains the same. If v_i is flipped, then it flips over the pocket's line of support, L, which is the perpendicular bisector of $v_i^k v_i^{k+1}$; see Figure 4. Because L supports the hull of P^k and x is inside P^k, x is on the same side of L as v_i^k. Thus $d(x, v_i^{k+1}) > d(x, v_i^k)$. This establishes that the distance from x to each vertex is a monotonically nondecreasing function of k.

Step 2. Next we argue that the sequence P^k approaches a limit polygon P^*. The perimeter of P^k is independent of k,

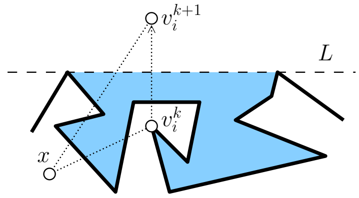

Figure 4: The distance from x to v_i increases by a flip.

for it is just the sum of the fixed edge lengths. The distance from x to v_i is bounded above by half the perimeter (because the polygon has to wrap around both x and v_i). Thus each distance sequence $d(x, v_i^k)$ has a limit. If we look at the distance sequences to v_i from three noncollinear points x_1, x_2, and x_3 inside P, their limits determine three circles (centered at x_1, x_2, and x_3) whose unique intersection point yields a limit position v_i^*. Then $P^* = \langle v_0^*, \ldots, v_{n-1}^* \rangle$.

Step 3. Let $\theta_i^k \in [0, 2\pi)$ be the directed angle $\angle v_{i-1}^k v_i^k v_{i+1}^k$. We observe that $\theta_i^k \in [\epsilon_i, 2\pi - \epsilon_i]$, where $\epsilon_i = \min\{\theta_i^k, 2\pi - \theta_i^k\}$. Indeed this relation holds for θ_i^0, and for θ_i^k to get closer to 0 or 2π, $d(v_{i-1}^k, v_{i+1}^k)$ would have to decrease, which is impossible by the distance argument detailed in Step 1. Because θ_i^k stays away from 0 and 2π, and the edge lengths of P^k are fixed, and therefore cannot approach zero, θ_i^k is a continuous function of the coordinates of the three vertices in P^k that define the angle. These vertices converge, so θ_i^k must also converge, and its limit is $\theta_i^* = \angle v_{i-1}^* v_i^* v_{i+1}^*$.

Step 4. We now distinguish between flat and nonflat vertices of P^*. A *flat vertex* v_i^* is one for which $\theta_i^* = \pi$. A *nonflat vertex* has $\theta_i^* \neq \pi$; it could be convex or reflex. Consider a vertex for which v_i^k moves an infinite number of times. We show that this vertex converges to a flat vertex v_i^* of P^*. Indeed, when v_i^k moves as a result of a pocket flip, $\theta_i^{k+1} = 2\pi - \theta_i^k$, as befalls any directed angle which is reflected. Consequently, there are infinitely many k for which $\theta_i^k \geq \pi$, and infinitely many for which $\theta_i^k \leq \pi$. Thus the limit θ_i^* can only be π.

Step 5. All that remains is to force a contradiction by showing that, once the nonflat vertices of P^* have been reached, no further flips are possible. Here we use the convex hull C^k of P^k, and the hull C^* of P^*. Of course, $P^k \subseteq C^k$ and $P^* \subseteq C^*$. We will obtain a contradiction to Fact A: for any k, $P^{k+1} \not\subseteq C^k$. The reason this fact holds is that, at every flip, the mirror image of the pocket area previously inside C^k is outside C^k. (See, for example, Figure 1: $P^1 \not\subseteq C^0$.)

Let $\bar{k}$ be a value of k for which only flat vertices of P^* have yet to converge. $P^{\bar{k}}$ includes all nonflat vertices in their final positions. Of course, P^* also includes all nonflat vertices in their final positions. Now, because the flat vertices of P^* cannot alter its hull beyond what the nonflat vertices already contribute, we know that $C^* \subseteq C^{\bar{k}}$. ($C^{\bar{k}}$ is conceivably a proper superset because of the vertices of P^k that are destined to be, but are not yet, flat vertices in P^*.)

Now consider $P^{\bar{k}+1}$. It is contained in all subsequent polygons and so in $P^* \subseteq C^*$. So we have reached Fact B: $P^{\bar{k}+1} \subseteq C^* \subseteq C^{\bar{k}}$. Fact B contradicts Fact A, so there cannot be an infinite sequence P^k. □

3.1 Discussion

To close, we outline some of the main differences between our proof and other arguments, particularly Kazarinoff and Bing's proof [8], which make exposition easier:

1. Whereas previous authors prove that P^k becomes constant after a finite number of steps, we prefer to reason by contradiction, proving that an infinite number of flips is impossible. This simplifies our reasoning. For example, it allows us to conclude that P^{k+1} always contains points not in C^k. Otherwise, this relation would only hold until the polygon has convexified.
2. We use directed angles instead of interior angles. This approach allows us to talk about limit angles without worrying about whether the limit polygon is simple.
3. We show that nonflat vertices of P^* converge in finite time. Kazarinoff and Bing [8] use vertices of C^* instead, and consider that all vertices of C^* are nonflat. Unfortunately, this view means that there can be fewer vertices in C^* than in P^k, so we lose the correspondence between vertices of C^* and vertices of P^k.

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