Fold-and-Cut Magic

Erik D. Demaine   Martin L. Demaine*

1 Fold-and-Cut Problem

Take a sheet of paper, fold it flat however you like, and make one complete straight cut. What are the possible shapes of the unfolded pieces? For example, Figure 1 shows how to produce a five-pointed star by folding and one straight cut. You could imagine cutting out the silhouette of your favorite animal, object, or geometric shape.

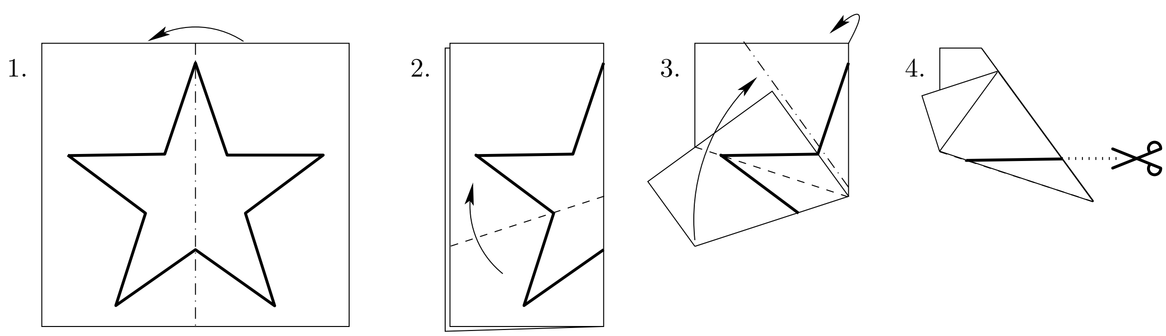

Figure 1: How to fold a square of paper so that one cut makes a five-pointed star.

Martin Gardner wrote about this *fold-and-cut* problem in a 1960 article in *Scientific American* called “Paper Cutting” [7]. In addition to providing some of the historical references below, he was the first to pose the then-open problem: What are the limits of this fold-and-cut process? What polygonal shapes can be cut out?

2 History

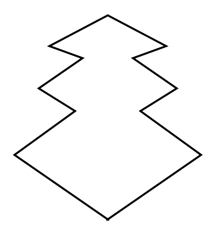

Figure 2: Japanese crest from [12].

The first published reference we know of is a Japanese puzzle book [12] by Kan Chu Sen in 1721. As one of many problems testing mathematical intelligence, this book asks the reader whether it is possible to fold a rectangular piece of paper flat and make one complete straight cut so as to produce a typical Japanese crest called *sangaibisi* (Figure 2), which translates to “three folded rhombics”. Towards the end, the book shows a solution.

Another early reference to folding and cutting is an 1873 article in *Harper's New Monthly Magazine* [1]. This article may be the first written account of a

*The Demaines are a father-and-son team. Erik D. Demaine, the son, is an assistant professor of computer science at MIT who likes anything algorithmic or geometric. Martin L. Demaine, the father, is a researcher in computer science at MIT who likes anything mathematical and artistic.

wide-spread tale of how Betsy Ross came to sew the first American flag. The story goes that, in 1777, George Washington and a committee of the Congress showed Betsy Ross plans for a flag with thirteen six-pointed stars, and asked her whether she could make such a flag. She said that she would be willing to try, but suggested that the stars should have five points. To support her argument, she demonstrated how easily a regular five-pointed star could be made by folding a sheet of paper flat and making one cut with scissors. The committee accepted her changes, and George Washington made a new drawing, which Betsy Ross followed to make the first American flag.

Folding and cutting has naturally also had its place in the magic community, and it is with these references that Martin Gardner was primarily familiar. Will Blyth was perhaps the first with his 1920 book *Paper Magic* [3], which includes a Maltese cross, a 6-pointed star, and an intricate multipiece altar. Harry Houdini was a general magician before he became the famous escape artist, and his 1922 book *Paper Magic* [8] describes a method for making a five-pointed star, similar to the one in Figure 1. This book may have in fact been ghost-written by fellow magician Walter Gibson [9]. Another magician, Gerald Loe, studied the fold-and-cut problem in some detail; his 1955 book *Paper Capers* [10] describes how to cut out arrangements of various geometric objects, such as a circular chain of stars. Gardner was particularly impressed with Loe's ability to cut out most letters of the alphabet. Gardner surveys a variety of other one-cut tricks in his *Encyclopedia of Impromptu Magic* [6].

3 Mathematics

Around the summer of 1997, Anna Lubiw and the present authors set out to answer Gardner's question from a computational geometry perspective: given a target shape, is there a good algorithm to determine whether it can be produced, and if so, to design a flat folding and say where to cut?

The surprising answer is that *every* polygonal shape can be produced. In fact, any desired pattern of cuts can be simultaneously produced, by folding and one straight cut. Such patterns allow us to make several shapes at once, to make holes within shapes, and to dissect the piece of paper into multiple adjoining shapes. Furthermore, there are algorithms that tell us where to fold and where to cut.

One solution and algorithm for the problem is by the original investigators [4, 5]. This solution is based on a structure called the "straight skeleton" which captures symmetries in the target pattern of cuts. Later on, Marshall Bern, David Eppstein, and Barry Hayes brought a new disk-packing approach to the problem, resulting in a second solution and algorithm for the problem [2].

These algorithms allow us to design many practical examples of the fold-and-cut idea that go far beyond what was previously possible. The first solution [4, 5] produces more practical foldings, so our examples in the next section use this algorithm.

Before we turn to examples, however, we should mention one minor catch in applying the general mathematical theory to practice, namely the precise definition of a "cut". In one model, cuts can be made along creases, effectively slitting the paper; in the other model, cuts cannot be made along creases, and must be slightly separated. The first model is more natural mathematically, but the second model is more practical.

The general result stated above applies only to the first model of cuts. For the second model, the patterns that can be produced by a single cut are precisely those where an even number of line segments meet at any point. All examples described in this article have this evenness property, and therefore obey the second, more practical model of cutting. More generally, this issue arises only for dissections; cutting out a single shape or even several disjoint shapes is not a problem because precisely two line segments meet at any point.

4 Examples

Figures 3, 4, 7, 8, and 9 show several examples of how to produce shapes by one straight cut.[1] All examples are based on the original algorithm [4, 5] The examples are designed to exploit symmetry and alignment to make them as easy as possible to fold. Naturally, some are easier than others.

The figures use the following notation. The shapes to be cut out are in thick lines; these lines are not creased. Dashed lines denote *valley* folds, where the paper is folded "towards" you. Dot-dashed lines denote *mountain* folds, where the paper is folded "away" from you. Some of the figures have a line of symmetry, in which case you must first fold along that line of symmetry, and then apply the remaining creases. In general, however, all creases must be folded more or less simultaneously.

The first example (Figure 3) produces the initials of the *Gathering for Gardner 5*. It also has 5 pieces, counting the exterior silhouette.

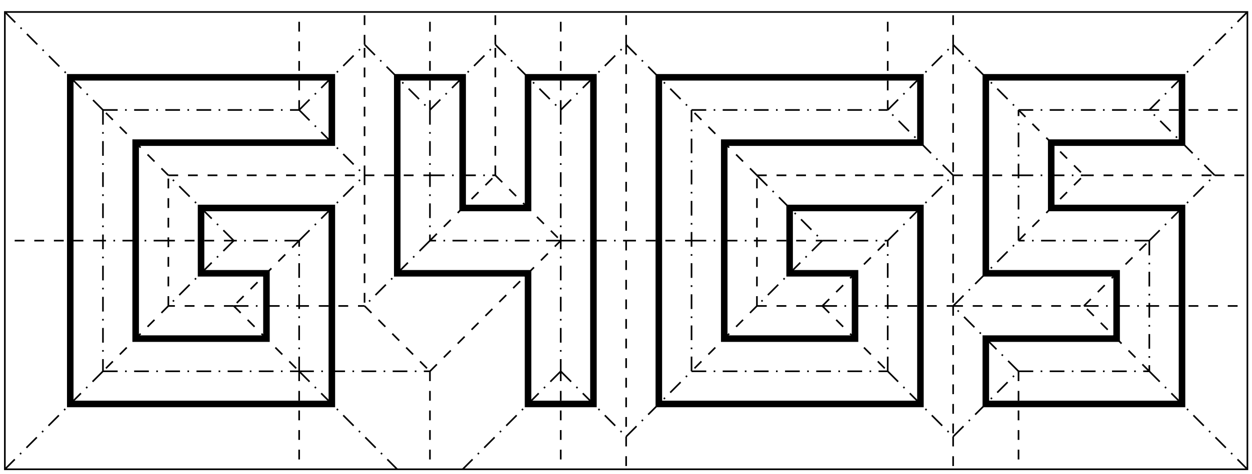

Figure 3: Initials of the Gathering for Gardner 5.

The second example (Figure 4) produces a cross shape along with the letters H-E-L-L, again consisting of 5 pieces. This example is a redesign of a classic fold-and-cut magic trick described by Gardner [6, 7] and shown in Figure 6. The classic trick is easier to fold, but requires the letters to be formed out of several smaller pieces. As a challenge, we considered what would be possible if we used an entire rectangle of paper to cut out precisely the four

[1] All foldings (except the classic in Figure 6) were designed by the present authors. Figures 7 and 9 have appeared before [4, 5].

letters and the cross. It is difficult to design a version with good proportions between the letters. For example, the more proportionate dissection in Figure 5 has an odd number of line segments coming together at some points, so it is unsuitable unless you allow cutting along the creases.

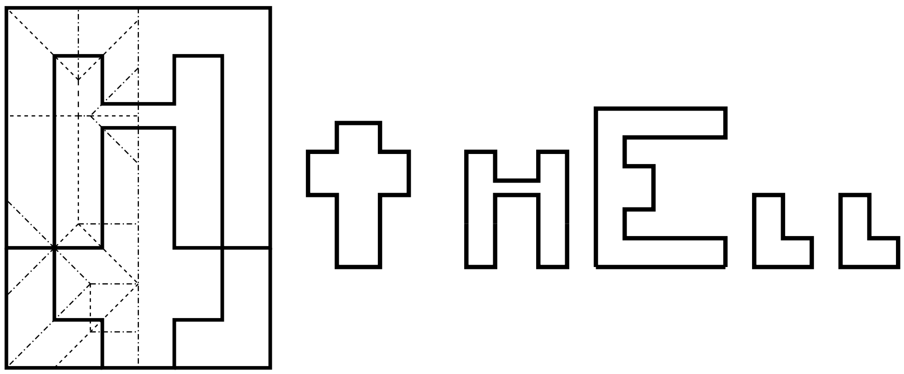

Figure 4: Cross and H-E-L-L. Fold in half first, then use the specified creases.

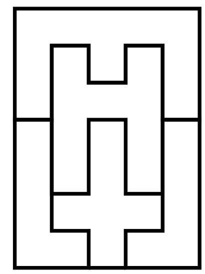

Figure 5: Unsuitable dissection.

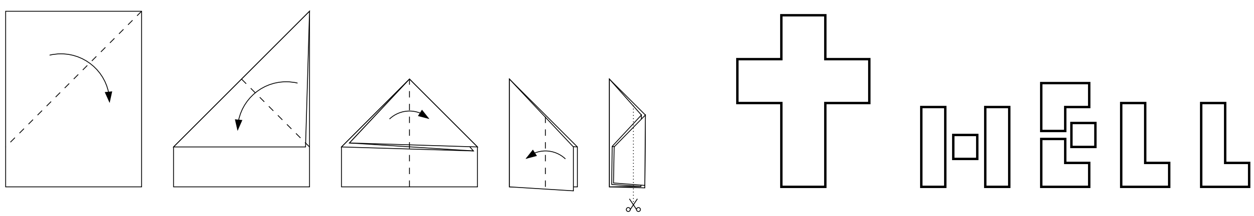

Figure 6: Classic method for producing cross and multipiece H-E-L-L.

For those unfamiliar with the cross/hell magic trick, the patter goes something like this. Two people, Good and Evil, arrive at the gates of heaven, but only Good has a ticket. Evil begs Good for help, so Good folds his ticket as shown, rips along a line, and hands Evil the smaller pieces. Unsure of what to do with the pieces, Evil hands them to St. Peter, who re-arranges them to spell H-E-L-L, to which Evil is appropriately directed. Good hands the remaining piece to St. Peter, who is pleased to unfold a cross.

The third example (Figure 7) consists of 5 triangles arranged in the pattern of a 5-pointed star [4, 5]. Gerald Loe [10] designed a similar example.

The fourth example (Figure 8) is a 5-piece dissection of the square whose pieces can be re-arranged to spell the word CUT. The underlying dissection was designed by Joseph Madachy [11].

The next three examples (Figure 9) are silhouettes of three animals: a swan, butterfly, and angelfish [4, 5].

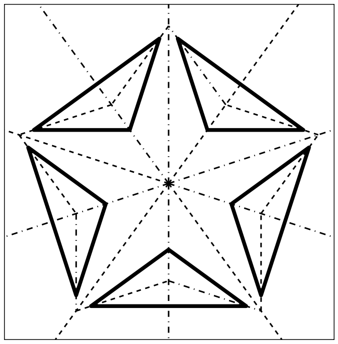

Figure 7: 5 triangles in a star pattern.

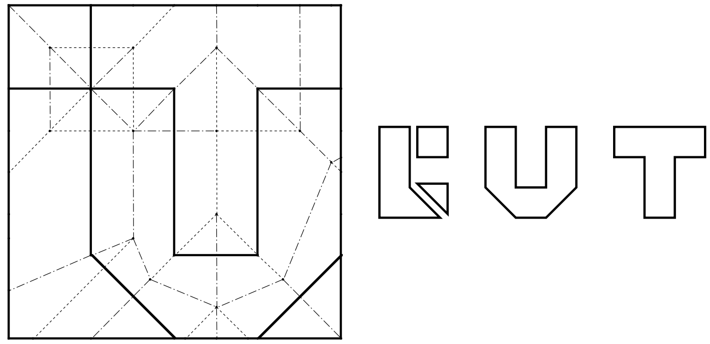

Figure 8: 5-piece dissection of C-U-T.

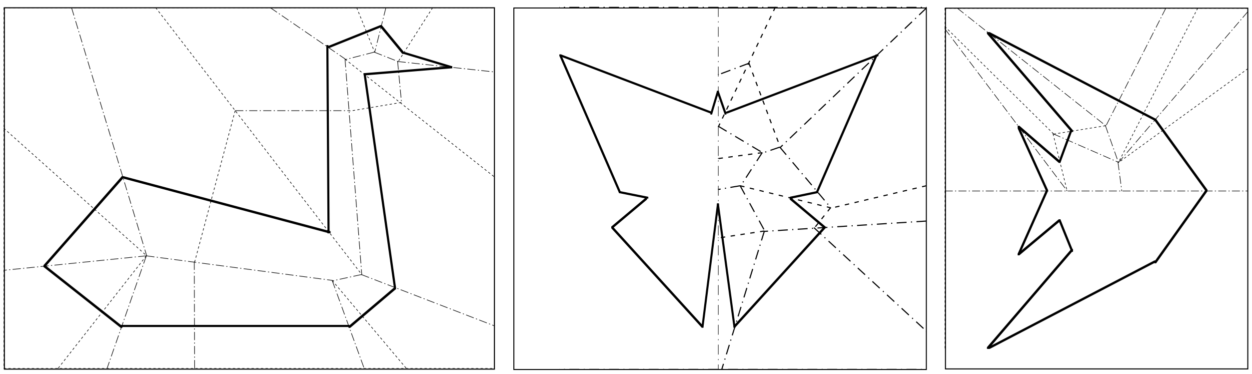

Figure 9: Swan, butterfly, and angelfish.

Acknowledgment

We thank Martin Gardner for inspiring us to work on this problem and thereby enter the area of computational origami.

References

[1] National standards and emblems. *Harper's New Monthly Magazine*, 47(278):171–181, July 1873.

[2] Marshall Bern, Erik Demaine, David Eppstein, and Barry Hayes. A disk-packing algorithm for an origami magic trick. In *Proceedings of the 3rd International Meeting of Origami Science, Math, and Education*, pages 17–28, Monterey, California, March 2001. Improvement of version appearing in *Proceedings of the International Conference on Fun with Algorithms*, Isola d'Elba, Italy, June 1998, pages 32–42.

[3] Will Blyth. Gains of the great war. In *Paper Magic: Tricks and Amusements with a Sheet of Paper*, pages 95–99. C. Arthur Pearson, Limited, 1920.

[4] Erik D. Demaine, Martin L. Demaine, and Anna Lubiw. Folding and cutting paper. In J. Akiyama, M. Kano, and M. Urabe, editors, *Revised Papers from the Japan Conference on Discrete and Computational Geometry*, volume 1763 of *Lecture Notes in Computer Science*, pages 104–117, Tokyo, Japan, December 1998.

[5] Erik D. Demaine, Martin L. Demaine, and Anna Lubiw. Folding and one straight cut suffice. In *Proceedings of the 10th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 891–892, Baltimore, MD, January 1999.

[6] Martin Gardner. Single-cut stunts. In *Encyclopedia of Impromptu Magic*, pages 424–428. Magic, Inc., 1978.

[7] Martin Gardner. Paper cutting. In *New Mathematical Diversions (Revised Edition)*, chapter 5, pages 58–69. The Mathematical Association of America, Washington, D.C., 1995. Appeared in *Scientific American*, June 1960.

[8] Harry Houdini. *Paper Magic*, pages 176–177. E. P. Dutton & Company, 1922. Reprinted by Magico Magazine.

[9] G. Legman. *Bibliography of Paper-Folding*. Priory Press, Malvern, England, 1952.

[10] Gerald M. Loe. *Paper Capers*. Magic, Inc., Chicago, 1955.

[11] Joseph S. Madachy. Geometric dissections. In *Madachy's Mathematical Recreations*, chapter 1, pages 15–33. Dover Publications, 1979. Reprint of *Mathematics on Vacation*, Scribner, 1975.

[12] Kan Chu Sen. *Wakoku Chiyekurabe (Mathematical Contests)*. 1721. Excerpts available from `http://theory.lcs.mit.edu/~edemaine/foldcut/sen_book.html`.