Graph Drawing '98

PLANAR DRAWINGS OF ORIGAMI POLYHEDRA

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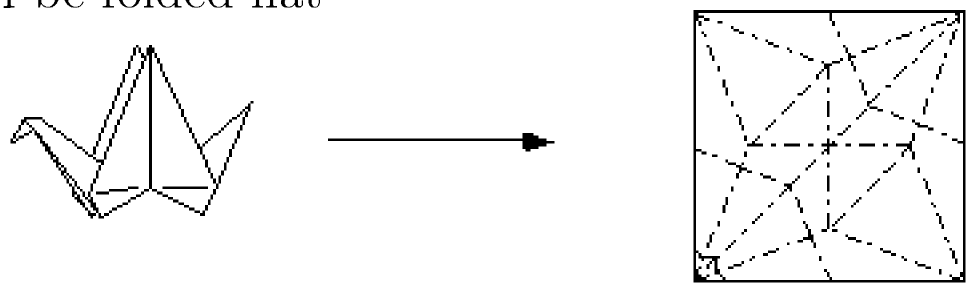
INTRODUCTION

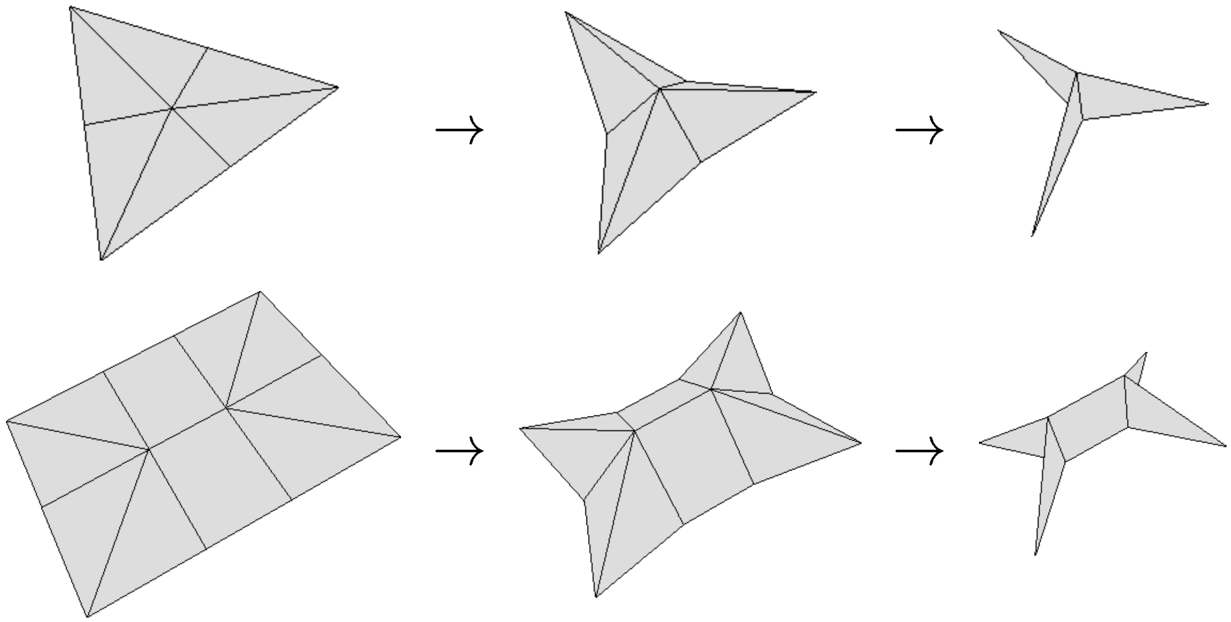
- Our goal is to use graph drawing to:
  - Discover fundamental properties of paper folding
  - Generally improve the understanding of origami mathematics
- One important problem in origami mathematics is to find a useful characterization of crease patterns that can be folded flat



  - This problem is NP-hard [Bern & Hayes, SODA'96]
  - However, this does not preclude simple characterizations, only their tractability
  - For example, one conjecture we have arrived at from this work is that every flat-foldable crease pattern has a Hamiltonian cycle
- Here we explore an infinite class of polyhedra, which arise from **origami bases**, by drawing their graphs
- We discover a recursive structure in the polyhedra, which was not apparent before and leads to interesting fractals

EXTREME BASE

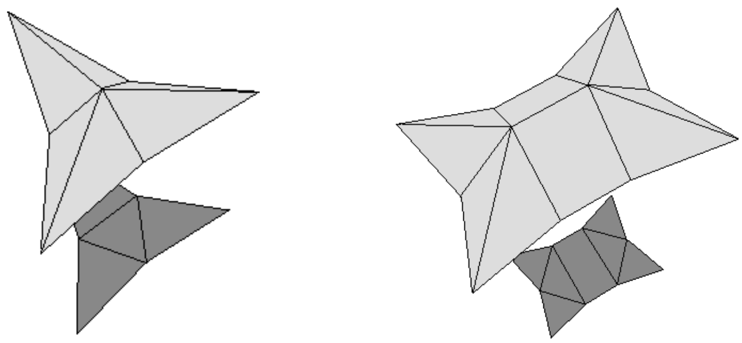
- An **extreme base** maps the boundary of a polygonal sheet of paper to a common plane
  - Here we just consider convex polygons
- Consists of two collections of folds:
  - The medial axis
  - Perpendicular folds
- Examples: Triangle and quadrangle



Unfolded — Partial folding — Folded

ORIGAMI POLYHEDRA

- To understand how the boundary of an extreme base becomes coplanar:
  - Take a partial folding of extreme base
  - Add the lower convex hull
- Examples:

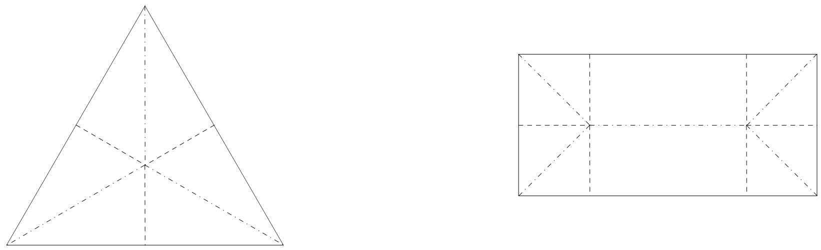


  - The lower convex hull is drawn below the partial folding in dark gray
- We call the result an **origami polyhedron**
- To gain insight into origami polyhedra and their underlying bases, we look at drawings of their graphs
  - Note that all polyhedra have planar straight-line drawings

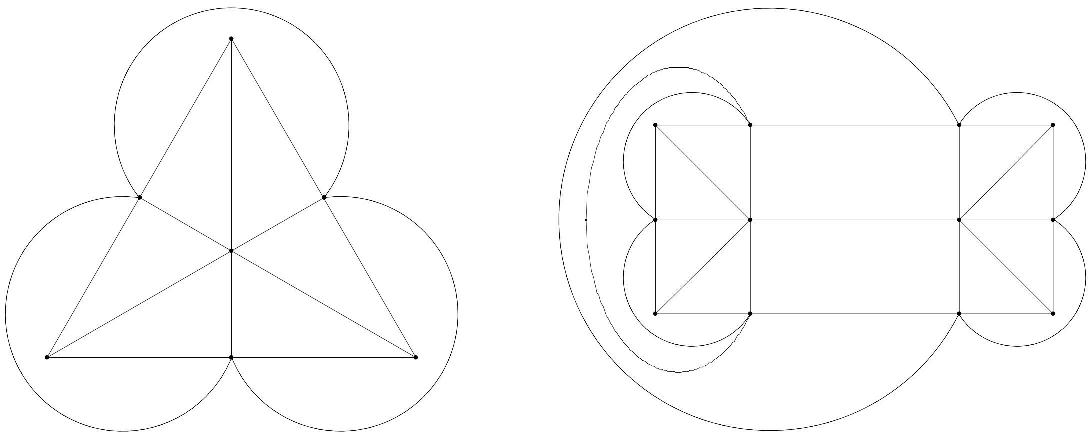
CURVED DRAWING

- A simple way to draw an origami polyhedron is the **curved drawing**

- Take the boundary of the paper
- Take the crease pattern for an extreme base



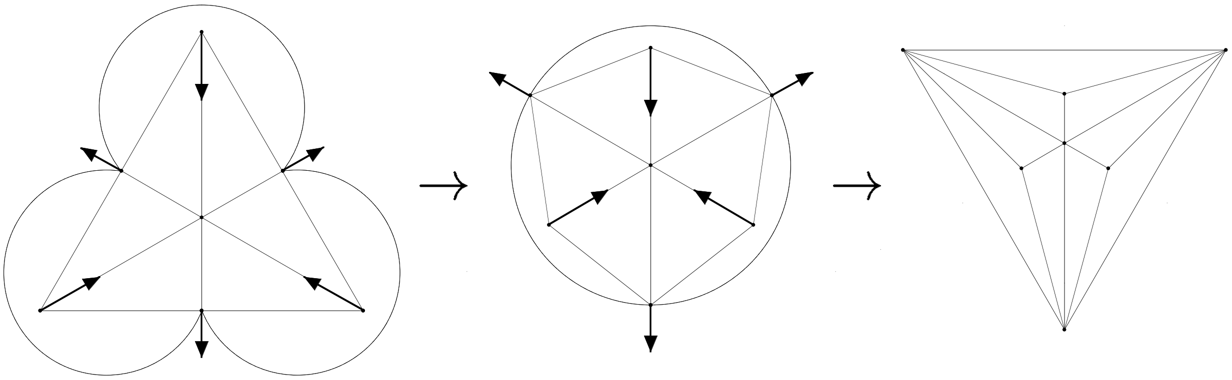
- Add curved edges to represent the lower convex hull
  - Cyclicly connect the ends of perpendiculars incident to a common medial-axis vertex



  - The nesting of the cycles is based on rooting the medial axis at some vertex

STRETCHING

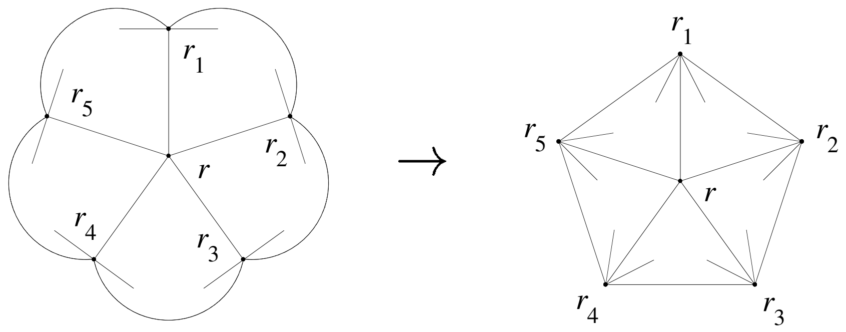
- To obtain a straight-line drawing, we can "stretch" the curved drawing
- For example, consider the triangle:



- A generalization of this idea is the **natural drawing**

NATURAL DRAWING

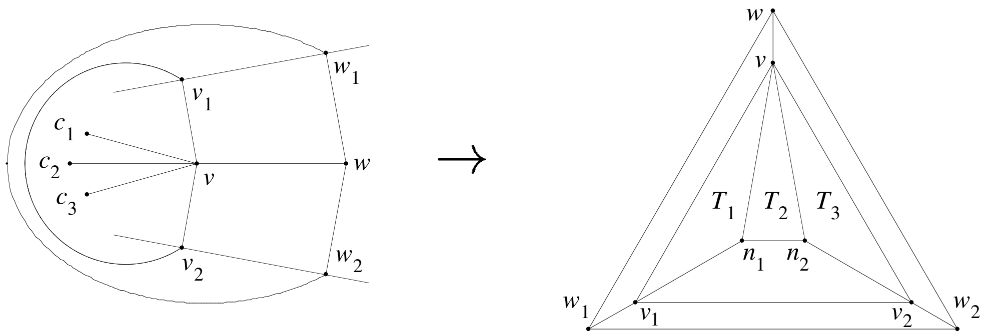
- Root the medial axis at some vertex r
  - This defines a parent map on the vertices of the medial axis
- Let $\deg(v)$ denote the degree of medial-axis vertex v
- Denote the ends of the perpendiculars incident to a medial-axis vertex v by $v_1, v_2, \ldots, v_{\deg(v)}$
- The outermost cycle is the one connecting the perpendicular ends $r_1, r_2, \ldots, r_{\deg(r)}$ around the root
  - Straighten this cycle as follows:



- Reserve triangle r_i, r_{i+1}, r in the straight drawing for child c_i of the root
  - See the next page for details

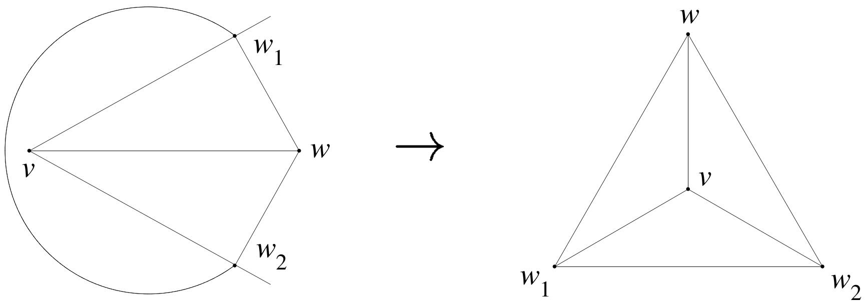
NATURAL DRAWING DETAILS

- Compute the natural drawing by recursing down the medial-axis tree
- In general, consider a vertex v of the medial axis whose parent is w
- Renumber their perpendicular ends so that v_1 is incident to w_1 and v_2 is incident to w_2
- Suppose that v has 3 children c_1, c_2, c_3
- Then we straighten as follows:



  - Triangle T_i is reserved for child c_i of vertex v

- The base case in this recursion is when v has no children, in which case we straighten as follows:

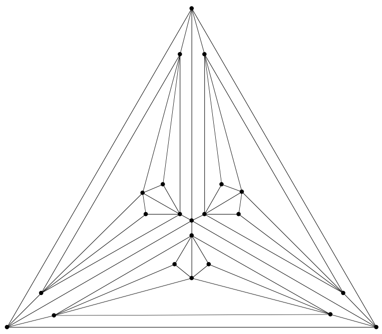


FRACTALS

- The natural drawing can be computed directly from the topology of the (rooted) medial axis
- Key advantage of the natural drawing:

  Illustrates the **recursive structure** of origami polyhedra

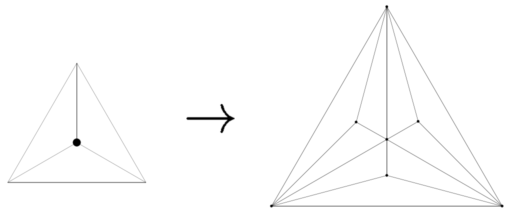
- We can make fractals using this concept
  - Lazily compute the topology of an infinite "medial axis"
  - For example, choosing every vertex to have degree 3 results in a self-similar drawing, which is approximately the following:



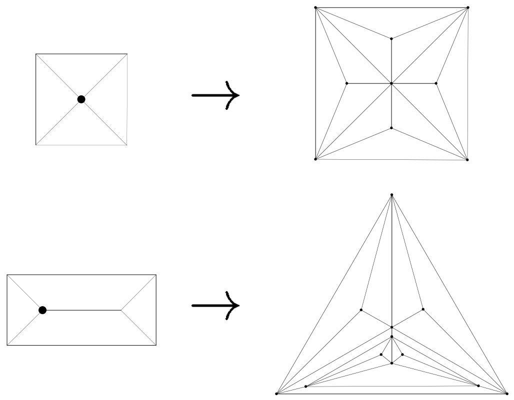
  - Alternatively, we can choose the degrees of the vertices randomly

Natural Drawings of Origami Polyhedra for Triangles through Hexagons

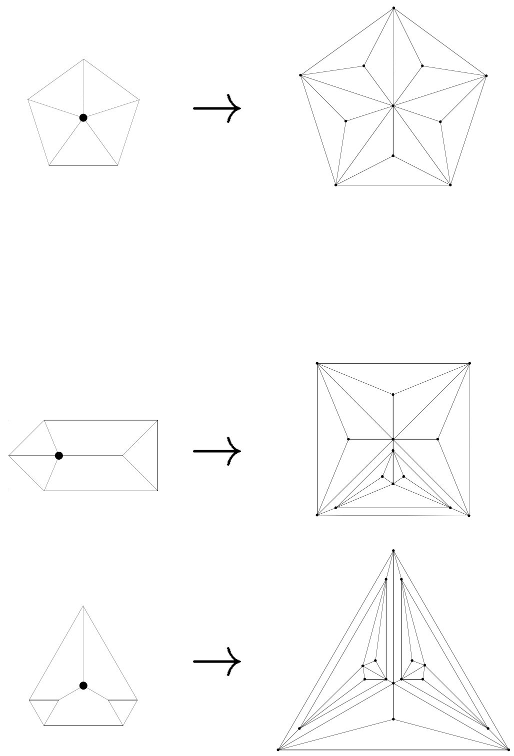
3 vertices



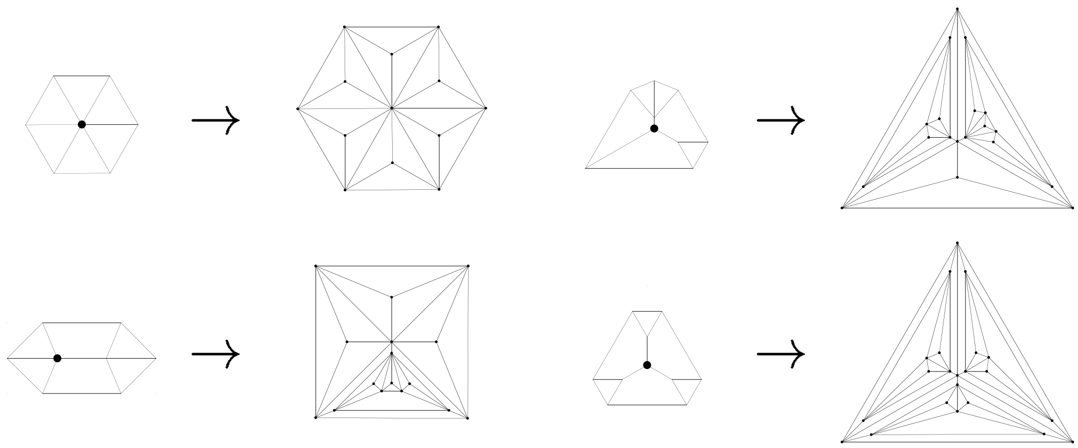
4 vertices



5 vertices

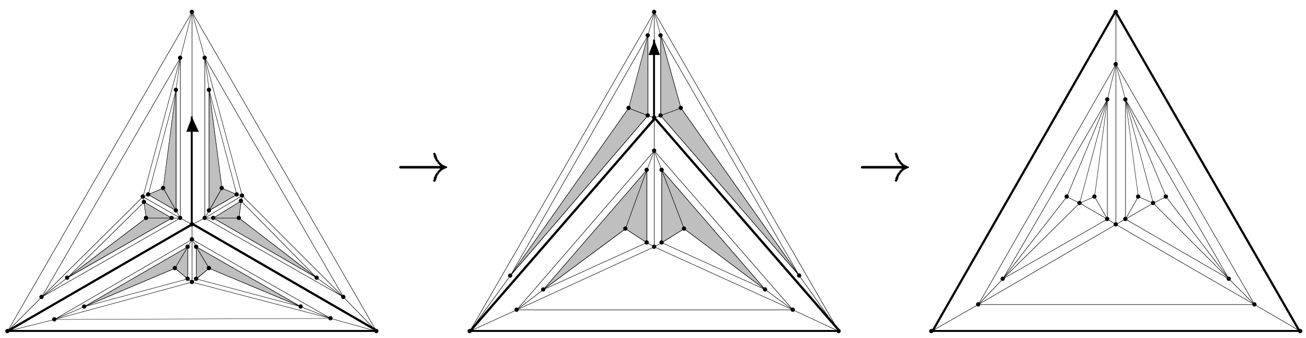


6 vertices



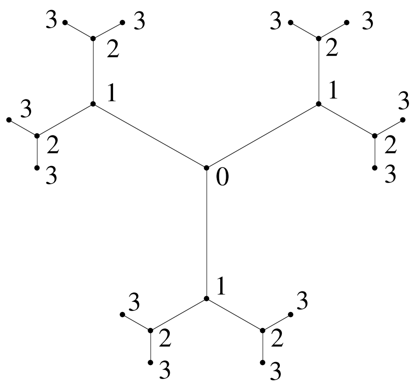
ZOOMING

- It is difficult to see the entire natural drawing when the medial-axis tree is deep
  - Formally, the angular resolution is $O(2^{-d})$ for a tree with depth d and degree-3 vertices
- We want to somehow view the graph, but preserve the recursive structure displayed by the natural drawing
- Solution: **Interactive zoom feature**
- Truncate the natural drawing after the largest viewable depth
  - Fill truncated regions in gray
- User can select which triangle in the next level to visit, or move up to previous level
  - Animation to zoom into the bottom triangle and make it equilateral:



TREE VIEW

- To avoid getting lost because of the stretching in the zoom feature, we also show a **tree view**
- The tree view displays the medial-axis tree with optimal angular resolution
  - In particular, it does not change shape during zooms



- The tree view scales to show a few levels above and below the current view
- Additional information in the tree view:
  - Highlight the current node
  - Highlight which nodes have been visited
  - Specify the node depths
  - Etc.