# Quickly deciding minor-closed parameters in general graphs

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**Abstract**

We construct algorithms for deciding essentially any minor-closed parameter, with explicit time bounds. This result strengthens previous results by Robertson and Seymour [1,2], Frick and Grohe [3], and Fellows and Langston [4] toward obtaining fixed-parameter algorithms for a general class of parameters.

## 1 Motivation

A major result from the seminal Graph Minors series of papers (in particular [1,2]) is that every minor-closed graph property is characterized by a finite set of forbidden minors. More precisely, for any property $P$ on graphs such that a graph having property $P$ implies that all its minors have property $P$, there is a finite set $\{H_1, H_2, \ldots, H_h\}$ of graphs such that a graph $G$ has property $P$ if and only if $G$ does not have $H_i$ as a minor for all $i = 1, 2, \ldots, h$. The algorithmic consequence of this result is that there exists an $O(n^3)$-time algorithm to decide any fixed minor-closed graph property, by finitely many calls to an $O(n^3)$-time minor test [1]. This consequence has been used to show the existence of polynomial-time algorithms for several graph problems, some of which were not previously known to be decidable [4].

It should be stressed that all of these algorithmic results (except the minor test) are nonconstructive: we know that efficient algorithms exist, but do not know what they are. The difficulty is that we know that a finite set of forbidden minors exists, but lack "a means of identifying the elements of the set, the cardinality of the set, or even the order of the largest graph in the set" [4]. Indeed, there is a mathematical sense in which any proof of the finite-forbidden-minors theorem must be nonconstructive [5].

A natural class of graph properties comes from graph parameters: a *parameter* assigns a nonnegative integer to every graph. Such a parameter defines an infinite sequence of properties, whether the graph has parameter value $\leq k$, for each $k \geq 0$. Parameters and their associated properties have been the subject of much study in *parameterized complexity* [6], a refinement of complexity theory where problems are augmented by parameters. In this subject, the main goal is to attain a *fixed-parameter algorithm*, that is, an algorithm whose running time is $f(k)\, n^{O(1)}$ where $k$ is the parameter value and $n$ is the problem size.

We can apply the graph-minor results to prove the existence of algorithms to compute parameters, provided the parameters are minor-closed. A parameter is *minor-closed* if its value never increases when taking a minor. Every property associated with a minor-closed parameter is also minor-closed. Therefore, for any fixed parameter and any fixed $k \geq 0$, there exists an $O(n^3)$-time algorithm that decides whether a graph has parameter value $\leq k$. Unfortunately, the existence of these algorithms does not necessarily imply the existence of a single fixed-parameter algorithm that works for all $k \geq 0$, because the algorithms for individual $k$ (in particular the set of forbidden minors) might be uncomputable. We do not even know an upper bound on the running time of these algorithms as a function of $n$ and $k$, because we do not know the dependence of the size of the forbidden minors on $k$.

In this paper we construct fixed-parameter algorithms for essentially all minor-closed parameters, with explicit time bounds in terms of $n$ and $k$. We require three trivial additional properties of the parameter: the parameter must be positive for some $g \times g$ grid, the parameter for a disconnected graph must be at least the sum of the parameter values for each connected component, and there must be an algorithm computing the parameter in $h(w)\, n^{O(1)}$ time for graphs of treewidth $w$. (See, e.g., [12] for a definition of treewidth.) These conditions are met by essentially all minor-closed parameters we have encountered; for example, all parameters whose corresponding properties can be expressed in monadic second-order logic satisfy the last condition [7].

The running time of our algorithm for computing such a parameter on general graphs is $\left[2^{2^{O(k^{2.5})}} + h(2^{O(k^{2.5})})\right] n^{O(1)}$. A conjecture of Robertson, Seymour, and Thomas [8] would improve this running time to $h(O(k \lg k))n^{O(1)}$, which is $2^{O(k \lg k)}n^{O(1)}$ for the typical case of $h(w) = 2^{O(w)}$. This conjectured time bound almost matches the fastest known fixed-parameter algorithms for several parameters, e.g., feedback vertex set, vertex cover, and a general family of vertex-removal problems [4]. Our result strengthens previous approaches of Robertson and Seymour [1,2], Frick and Grohe [3], and Fellows and Langston [4] to obtaining fixed-parameter algorithms for a general class of parameters.

## 2 Main Result

Our result is based on the following theorem of Robertson, Seymour, and Thomas [8]:

**Theorem 1** [8] *Every graph of treewidth larger than $20^{2r^5}$ has an $r \times r$ grid as a minor.*

Our main result is as follows:

**Theorem 2** *Consider a minor-closed parameter $P$ that is positive on some $g \times g$ grid, is at least the sum over the connected components of a disconnected graph, and can be computed in $h(w)\, n^{O(1)}$ time given a width-$w$ tree decomposition of the graph. Then there is an algorithm that decides whether $P$ is at most $k$ on a graph* *with $n$ vertices in $\left[2^{2^{O(g\sqrt{k})^5}} + h(2^{O(g\sqrt{k})^5})\right] n^{O(1)}$ time.*

**Proof.** First we claim that if the parameter $P$ has value $k$ on some graph $G$, then the treewidth of $G$ is at most $20^{2(g\sqrt{k}+g)^5}$. Suppose to the contrary that the treewidth of $G$ is larger. Then by Theorem 1, $G$ has an $r\times r$ grid as a minor where $r \geq g(\sqrt{k}+1)$. By cutting edges of this $r \times r$ grid, we can obtain a disjoint union of $\lfloor r/g \rfloor^2$ copies of the $g \times g$ grid. Therefore this disjoint union of grids is also a minor of $G$. Because the parameter is minor-closed, its value on this disjoint union is a lower bound on its value on $G$. The parameter value on the disjoint union is at least the sum of the parameter value on each of the $g \times g$ grids, each of which is at least 1. Therefore $P(G)$ is at least $\lfloor r/g \rfloor^2 > k$, a contradiction.

The algorithm is as follows. We use as a subroutine Amir's algorithm [9] (or Robertson and Seymour's algorithm [10]) which, for a given graph $G$ and integer $\omega$, either reports that the treewidth of $G$ is more than $\omega$, or produces a tree decomposition of width at most $(3 + \frac{2}{3})\omega$, in $O(2^{3.698\omega} n^{3+\epsilon})$ time for any $\epsilon > 0$. Letting $\omega = 20^{2(g\sqrt{k}+g)^5}$, we either find a tree decomposition of width $w = O(\omega)$, or we determine that the treewidth is more than $20^{2(g\sqrt{k}+g)^5}$, in which case we know that the parameter value is more than $k$ and the algorithm can report "no". In the first case, we run the $h(w)\, n^{O(1)}$ algorithm using the computed tree decomposition, and output whether the answer is at most $k$. The total running time is $\left[2^{O(20^{2(g\sqrt{k}+g)^5})} + h(O(20^{2(g\sqrt{k}+g)^5}))\right] n^{O(1)}$. □

Improvements to the bound in Theorem 1 translate directly into improvements to our time bound. Robertson, Seymour, and Thomas [8] have proved that some graphs have treewidth $\Omega(r^2 \lg r)$ but have grid minors only of size $O(r) \times O(r)$, so a bound better than $\Theta(r^2 \lg r)$ is not possible. They conjecture that the correct bound is indeed $\Theta(r^2 \lg r)$. This conjecture would have the following consequence:

**Theorem 3** *Assume that every graph of treewidth larger than $\Theta(r^2 \lg r)$ has an $r \times r$ grid as a minor. Then for every minor-closed parameter $P$ satisfying the conditions of Theorem 2, there is an algorithm that decides whether $P$ is at most $k$ on any graph in $\left[2^{O(g^2 k \lg(gk))} + h(O(g^2 k \lg(gk)))\right] n^{O(1)}$ time.*

Our result is in some sense a generalization of minor-bidimensionality from minor-closed graph families to general graphs. Bidimensional parameters are a broad family of graph parameters introduced in a series of papers [11,12,13]. A challenging open question is whether the results of this paper can be generalized to contraction-bidimensional parameters, which include e.g. many domination-type parameters. The difficulty is that the parameter is closed under contractions but not minors: the parameter may increase from a vertex or edge deletion.

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