Hinged Dissection of Polypolyhedra

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1 Introduction. A *dissection* of two figures (solid 2D or 3D shapes, e.g., polygons or polyhedra) is a way to cut the first figure into finitely many (compact) pieces and to rigidly move those pieces to form the second figure. It is well-known that any two polygons of the same area have a dissection, but not every two polyhedra of the same volume have a dissection [3, 5].

A *hinged dissection* of two figures is a dissection in which the pieces are hinged together at points (in 2D or 3D) or along edges (in 3D), and there is a motion between the two figures that adheres to the hinging, keeping the hinge connections between pieces intact. While a few hinged dissections such as the one in Figure 1 are quite old (1902), hinged dissections have received most of their study in the last few years; see [6, 4]. It remains open whether every two polygons of the same area have a hinged dissection, or whether every two polyhedra that have a dissection also have a hinged dissection.

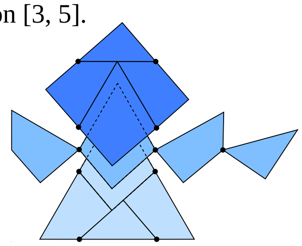

Figure 1: Hinged dissection of square and equilateral triangle, described by Dudeney [5].

2 Results. In this paper we develop a broad family of 3D hinged dissections for a class of polyhedra called polypolyhedra. For a polyhedron P with labeled faces, a *polypolyhedron of type* P is an interior-connected non-self-intersecting solid formed by joining several rigid copies of P wholly along identically labeled faces. (Such joinings are possible only for reflectionally symmetric faces.) Figure 2 shows two *polycubes* (where P is a cube).

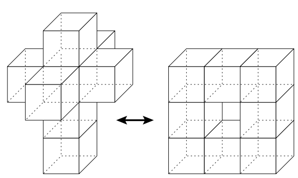

Figure 2: Two polycubes of order 8, which have a 24-piece edge-hinged dissection by our results.

For every polyhedron P and positive integer n, we develop one hinged dissection that folds into all (exponentially many) n-polyhedra of type P. The number of pieces in the hinged dissection is linear in n and the combinatorial complexity of P. For polyplatonics, we give particularly efficient hinged dissections, tuning the number of pieces to the minimum possible among a natural class of "regular" hinged dissections of polypolyhedra. For polyparallelepipeds (where P is any fixed parallelepiped), we give hinged dissections in which every piece is a scaled copy of P. All of our hinged dissections are hinged along edges and form a cyclic chain of pieces, which can be broken into a linear chain of pieces.

Our results generalize analogous results about hinged dissections of "polyforms" in 2D [4].

Like most previous theoretical work in hinged dissections, we do not know whether our hinged dissections can be folded from one configuration to another without self-intersection. However, we prove the existence of such motions for the most complicated gadget, the twister.

3 Proof Overview. Our construction of a hinged dissection of all n-polyhedra of type P divides into two parts. First, we find a suitable hinged dissection of the base polyhedron P. The exact constraints on this dissection vary, but two necessary properties are that the hinged dissection must be (1) cyclic, forming a closed chain of pieces, and (2) *exposed* in the sense that, for every face of P, there is a hinge in H that lies on the face (either interior to the face or on its boundary). For platonic solids, these hinges will be edges of the polyhedron; in the general case, we place these hinges along faces' lines of reflectional symmetry. Second, we repeat n copies of this hinged dissection of P, spliced together into one long closed chain. Finally, we prove that this new hinged dissection can fold into all n-polyhedra of type P, by induction on n.

3.1 Platonic Solids. Figure 3 shows an exposed cyclic hinged dissection of each of the platonic solids. Basically, each piece comes from carving the k-sided platonic solid into k face-based pyramids with the platonic solid's centroid as the apex. As drawn, these hinged dissections consist of k pieces, but by merging consecutive pairs of pieces along their common face, the number of pieces can be reduced to $k/2$ pieces while maintaining exposed hinges. These exposed hinged dissections have the fewest possible pieces, subject to the exposure constraint, because a hinge can simultaneously satisfy at most two faces of the original polyhedron.

3.2 General Case. In the general case, we use a 3D generalization of the straight skeleton [1] to decompose a given polyhedron into a collection of cells, exactly one cell per facet, such that exactly one cell is incident to each facet. These cells form the pieces in an exposed hinged dissection. For these pieces can be connected together into a cyclic hinged dissection, we need to first arrange for the polyhedron P to have a Hamiltonian dual graph.

In fact, we make two main modifications to P's sur-

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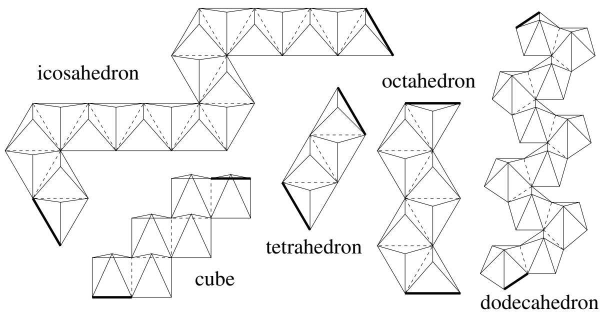

Figure 3: Unfolded exposed cyclic hinged dissections of the platonic solids. The bold lines indicate a pair of edges that are joined by a hinge but have been separated in this figure to permit unfolding. The dashed lines denote all other hinges between pieces. In the unfolding, the bases of all of the pyramid pieces lie on a plane, and the apexes lie above that plane (closer to the viewer).

face. First, we divide each reflectionally symmetric face of P along one of its lines of symmetry, producing a polyhedron P'. Second, we divide each face of P' so that any spanning tree of the faces in P' translates into a Hamiltonian cycle in the resulting polyhedron P''. This reduction is similar to the Hamiltonian triangulation result of [2] as well as a refinement for hinged dissection of 2D polyforms [4, Section 6]. We conceptually triangulate each face f of P' using chords (though we do not cut along the edges of that triangulation). Then, for each triangle, we cut from an arbitrarily chosen interior point to the midpoints of the three edges. Figure 4 shows an example. For any spanning tree of the faces of P', we can walk around the tree (follow an Eulerian tour) and produce a Hamiltonian cycle on the faces of P''.

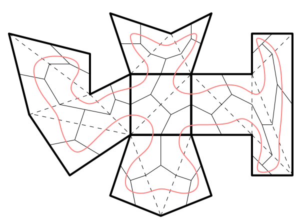

Figure 4: Hamiltonian refinement of five faces in a hypothetical polyhedron.

In particular, we can start from the matching on the faces of P' from the reflectionally symmetric pairing, and choose a spanning tree on the faces of P' that contains this matching. Then the resulting Hamiltonian cycle in P'' crosses a subdivided edge of every line of symmetry. (In fact, the Hamiltonian cycle crosses every subdivided edge of every line of symmetry.) Thus, in the exposed cyclic hinged dissection of the Hamiltonian polyhedron P'', there is an exposed hinge along every line of symmetry. Therefore all joinings between copies of P'' can use these hinges.

3.3 Putting Pieces Together. We use induction to prove that the nth repetition of the exposed cyclic hinged dissection of P described above can fold into any n-polyhedron of type P. The base case of $n = 1$ is trivial. Given an n-polyhedron Q of type P, one copy P_1 of P can be removed to produce an $(n-1)$-polyhedron Q'. By induction, the $(n-1)$st repetition of the exposed hinged dissection can fold into Q'. Also, P_1 itself can be decomposed into an instance of the exposed hinged dissection. Our goal is to merge these two hinged dissections. Essentially, we show that the hinged dissections can be placed against the shared face between P_1 and Q' in such a way that (1) a hinge of the exposed hinged dissection of P_1 coincides with a hinge of the hinged dissection of Q', and (2) the four pieces involved in these two hinges can be rehinged so that all pieces are connected in a single cycle, and that cycle is exactly the nth repetition of the exposed hinged dissection of P.

3.4 Mutually Rotated Base Polyhedra: Twisters. If a face is k-fold symmetric for $k \geq 3$, then there are several ways to glue two copies of P along this face. These different gluings produce different polypolyhedra if P itself is not k-fold symmetric. However, only one of the gluings can be produced by the inductive argument described above, because only one relative rotation will align the hinges that lie along the one chosen line of symmetry.

To enable these kinds of joinings, we embed the *twister gadget* shown in Figure 5 beneath each face of P'' that has k-fold symmetry for $k \geq 3$. This gadget consists of $8k$ cyclically hinged pieces that allow any integer multiple of $1/k$ rotation of one set of pieces with respect to the other pieces.

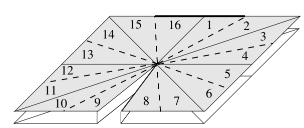

Figure 5: This 32-piece twister gadget allows turns of one-quarter of a twist. Although the pieces look two dimensional, they have thickness (they are prisms). The gaps between pieces 8 and 9 in subfigure (a) and between the top and bottom layers are for visual clarity only; in fact, the two layers are flush. Solid segments denote lengthwise hinges on the "inside" layer; dashed segments denote tiny hinges on the perimeter.

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