Hinged Dissection of Polypolyhedra

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Abstract. This paper presents a general family of 3D hinged dissections for *polypolyhedra*, i.e., connected 3D solids formed by joining several rigid copies of the same polyhedron along identical faces. (Such joinings are possible only for reflectionally symmetric faces.) Each hinged dissection consists of a linear number of solid polyhedral pieces hinged along their edges to form a flexible closed chain (cycle). For each base polyhedron P and each positive integer n, a single hinged dissection has folded configurations corresponding to all possible polypolyhedra formed by joining n copies of the polyhedron P. In particular, these results settle the open problem posed in [7] about the special case of polycubes (where P is a cube) and extend analogous results from 2D [7]. Along the way, we present hinged dissections for polyplatonics (where P is a platonic solid) that are particularly efficient: among a type of hinged dissection, they use the fewest possible pieces.

1 Introduction

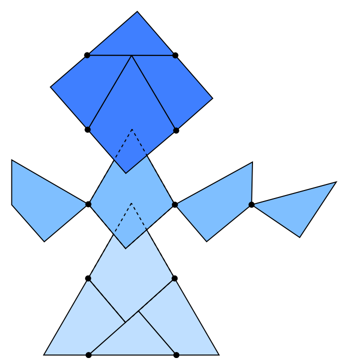

Fig. 1. Hinged dissection of square and equilateral triangle [8]. Different shades show different folded states.

A *dissection* of a set of figures (solid 2D or 3D shapes, e.g., polygons or polyhedra) is a way to cut one of the figures into finitely many (compact) pieces such that it can be transformed into any other of the figures by moving the pieces rigidly. Dissections have been studied extensively, particularly in 2D [12, 15]. It is well-known that any two polygons of the same area have a dissection [5, 12, 16]. By transitivity, it is easy to extend this result to a dissection of any finite set of polygons. Thus, in this context, the main interest is in finding the

* Supported in part by NSF CAREER award CCF-0347776 and DOE grant DE-FG02-04ER25647.
** Work begun when author was at Tufts University. Supported in part by NSF grant EIA-99-96237.
*** Supported in part by NSF grants EIA-99-96237 and CCF-0431027.

dissection of the polygons that uses the fewest possible pieces. On the other hand, not every two polyhedra of the same volume have a dissection: for example, there is no dissection of a regular tetrahedron and an equal-volume cube [5]. This result was a solution to Hilbert's Third Problem [5].

A *hinged dissection* of a set of figures is a dissection in which the pieces are hinged together at points (in 2D or 3D) or along edges (in 3D), and there is a motion between any two of the figures that adheres to the hinging, keeping the hinge connections between pieces intact. While a few hinged dissections such as the one in Figure 1 are quite old [8], hinged dissections have received most of their study in the last few years [3, 7, 9, 13]. It remains open whether every two polygons of the same area have a hinged dissection, or whether every two polyhedra that have a dissection also have a hinged dissection. It also remains open whether hinge-dissectability is transitive.

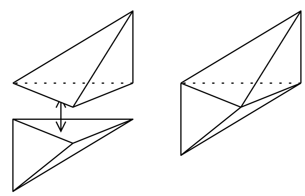

Fig. 2. Joining two rigid copies of a tetrahedron. The face of joining is reflectionally symmetric.

In this paper we develop a broad family of 3D hinged dissections for a class of polyhedra called polypolyhedra. For a polyhedron P with labeled faces, a *polypolyhedron of type* P is an interior-connected non-self-intersecting solid formed by joining several rigid copies of P wholly along identically labeled faces. See Figure 2. These joinings must perfectly match two opposite orientations of the same face of P, so joinings can occur only along faces with reflectional symmetry. We call P the *base polyhedron*. If a polypolyhedron consists of n rigid copies of P, we call it an *n-polyhedron of type P*. Examples of polypolyhedra include *polycubes* (where P is a cube) or more generally *polyplatonics* (where P is any fixed platonic solid); in any of these cases, any pair of faces can be joined because of the regular symmetry of the platonic solids. See Figure 3 for some examples of polycubes.

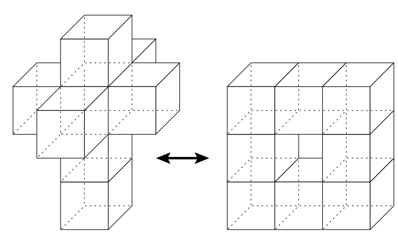

Fig. 3. Two polycubes of order 8, which have a 24-piece edge-hinged dissection by our results.

For every polyhedron P and positive integer n, we develop one hinged dissection that folds into all (exponentially many) n-polyhedra of type P. This result is superior to having one hinged dissection between every pair of n-polyhedra of type P. The number of pieces in the hinged dissection is linear in n and the combinatorial complexity of P. For polyplatonics, we give particularly efficient hinged dissections, tuning the number of pieces to the minimum possible among a natural class of "regular" hinged dissections of polypolyhedra. For polyparallelepipeds (where P is any fixed parallelepiped), we give hinged dissections in which every piece is a scaled copy of P. All of our hinged dissections are hinged along edges and form a cyclic chain of pieces, which can be broken into a linear chain of pieces.

Our solution combines several techniques to obtain increasingly more general families of hinged dissections. We reduce the problem of finding a hinged dissection of polypolyhedra of type P to finding a hinged dissection of P that has "exposed hinges" at certain locations on its surface. We find the first such hinged dissection for every platonic solid, exploiting that such a solid is star-shaped and has a Hamiltonian cycle on its faces. Then we relax the star-shaped constraint, generalizing P to be any solid with a Hamiltonian cycle on its faces, using a more general refinement scheme based on the straight skeleton. Then we relax the Hamiltonicity constraint by using a Hamiltonian refinement scheme. Finally, we show how faces with more than a single reflectional symmetry can be joined even when their labeled rotations are not equal. This step uses a general "twister" gadget, a hinged dissection that can rotate by any angle that is a multiple of $360°/k$ for fixed k.

Our results generalize analogous results about hinged dissections of "polyforms" in 2D [7]. For a polygon P with labeled edges, a *polyform of type P* is an interior-connected non-self-intersecting planar region formed by joining several rigid copies of P wholly along identically labeled edges. In particular, polyforms include polyominoes (where P is a square) and polyiamonds (where P is an equilateral triangle). In 2D, edges are always reflectionally symmetric (about their midpoint), so a polyform can join any pair of identically labeled edges. For any polygon P and positive integer n, [7] develops a single vertex-hinged dissection that folds into all n-forms of type P. The same paper asks whether analogous dissections exist in 3D, in particular for polycubes; we solve this open problem, building on the general inductive approach of [7].

We do not know whether our hinged dissections can be folded from one configuration to another without self-intersection. (The same is true of most previous theoretical work in hinged dissections [3, 7, 9].) However, we demonstrate such motions for the most complicated gadget, the twister.

Our results have applications in self-assembly and nanomanufacturing, and may find applications in self-reconfigurable robotics. Existing reconfigurable robots (see, e.g., [19]) consist of units that can attach and detach from each other, and this mechanism is complicated; 3D hinged dissection may offer a way to avoid this complication and still achieve arbitrary reconfiguration.[4] In self-assembly, recent progress has enabled chemists to build millimeter-scale "self-working" 2D hinged dissections [17]. An analog for 3D hinged dissections may enable building a complex 3D structure out of a chain of units. If the process is programmable, we could even envision an object that can re-assemble itself into different 3D structures on demand. These directions have recently been explored (so far at a more macroscale) using ideas from this paper [14].

2 Polyplatonics

In this section we demonstrate our approach for constructing a hinged dissection of polypolyhedra of type P in the special case that P is a platonic solid. Although

[4] This idea was suggested by Joseph O'Rourke in personal communication, Nov. 2004.

several of the details change in more general settings in later sections, the overall approach remains the same.

First, we find a suitable hinged dissection of the base polyhedron P. The exact constraints on this dissection vary, but two necessary properties are that the hinged dissection must be (1) *cyclic*, forming a closed chain (cycle) of pieces in which there is a single hinge connecting every consecutive pair of pieces and there are no other hinges, and (2) *exposed* in the sense that, for every face of P, there is a hinge in H that lies on the face (either interior to the face or on its boundary). For platonic solids, these hinges will be edges of the polyhedron. Second, we repeat n copies of this hinged dissection of P, spliced together into one long closed chain. Finally, we prove that this new hinged dissection can fold into all n-polyhedra of type P, by induction on n.

2.1 Exposed Cyclic Hinged Dissections of Platonic Solids

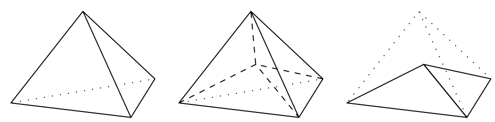

Fig. 4. Carving a regular tetrahedron into four face-based pyramids.

We construct an exposed cyclic hinged dissection of any platonic solid as follows. First we carve the platonic solid into a cycle of face-based pyramids with the platonic solid's centroid as the apex. Thus, a refined tetrahedron consists of four triangle-based pyramids (irregular tetrahedra); a refined cube consists of six square-based pyramids; a refined octahedron consists of eight triangle-based pyramids; a refined dodecahedron consists of twelve pentagon-based pyramids; and a refined icosahedron consists of twenty triangle-based pyramids. Every platonic solid has a Hamiltonian cycle on its faces. Consequently, the pieces in the refinement can be hinged together in a cycle, following the Hamiltonian path on the faces. Figure 5 shows unfoldings of these hinged dissections, in particular illustrating the Hamiltonian cycle.

Because there is a hinge dual to every edge in the Hamiltonian path on the faces, every face of the platonic solid has exactly two hinges. Therefore, the hinged dissection is exposed. Even more, we can merge adjacent pairs of pyramids along a face, halving the number of pieces, and leave exactly one hinge per face of the platonic solid. Now two faces share every hinge, but still the hinged dissection is exposed because every face has a hinge along its boundary. Thus we have proved

Theorem 1. *The platonic solid with f faces has an exposed cyclic hinged dissection of $f/2$ pieces in which every hinge is an edge of the platonic solid.*

These exposed hinged dissections have the fewest possible pieces, subject to the exposure constraint, because a hinge can simultaneously satisfy at most two faces of the original polyhedron.

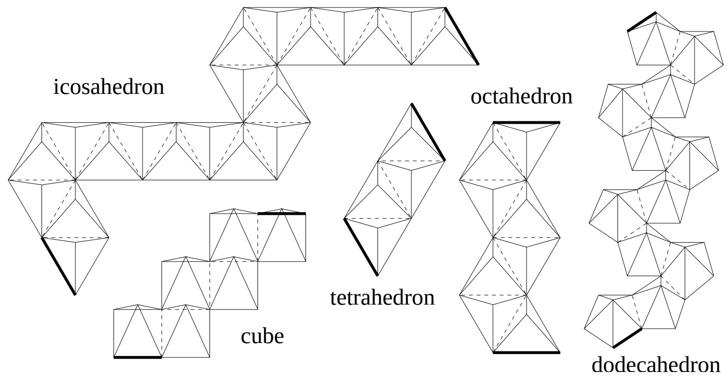

Fig. 5. Unfolded exposed cyclic hinged dissections of the platonic solids. The bold lines indicate a pair of edges that are joined by a hinge but have been separated in this figure to permit unfolding. The dashed lines denote all other hinges between pieces. In the unfolding, the bases of all of the pyramid pieces lie on a plane, and the apexes lie above that plane (closer to the viewer).

2.2 Inductive Hinged Dissection

Next we show how to build a hinged dissection of all n-platonics of type P based on a repeatable hinged dissection of a platonic solid P. The hinged dissection is essentially n repetitions of the exposed hinged dissection from the previous section. Specifically, the *nth repetition* of a cyclic hinged dissection is the result of cutting the cyclic hinged dissection at an arbitrary hinge to form an open chain, repeating this open chain n times, and then reconnecting the ends to restore a closed chain. Thus, if there are k pieces $H_1, H_2, \ldots, H_k$ connected in that order (and cyclically) in a cyclic hinged dissection, then the nth repetition consists of nk pieces $H_1, \ldots, H_k, H_1, \ldots, H_k, \ldots\ldots H_1, \ldots, H_k$ connected in that order (and cyclically). (Although the order $H_1, \ldots, H_k$ depends on where we cut the cyclic order, the resulting nth repetition is independent of this cut.)

We prove that this hinged dissection has the desired foldings by an inductive/incremental construction based on the following tool, similar to [7, Prop. 1]:

Lemma 1. *Every n-polyhedron of type P has a copy of P whose removal results in a (connected) $(n-1)$-polyhedron, provided $n > 1$.*

Proof. The graph of adjacencies between copies of P in an n-polyhedron is a connected graph on n vertices. Any spanning tree of this graph has at least two leaves, and the removal of either leaf leaves the original graph connected. The resulting pruned graph is the adjacency graph of a $(n-1)$-polyhedron. □

Reversing the inductive process of this lemma implies that any n-polyhedron of P can be built up by adding one copy of P at a time, yielding a connected 1-, 2-, …, and $(n-1)$-polyhedron along the way.

Theorem 2. *Given an exposed cyclic hinged dissection of the platonic solid P in which exactly one piece is incident to each face of P, the nth repetition of this hinged dissection can fold into any n-platonic of type P.*

Proof. The proof is by induction. The base case of $n = 1$ is trivial: there is only one 1-platonic of type P, namely P itself. The exposed hinged dissection satisfies all the desired properties.

Consider an n-platonic Q of type P. By Lemma 1, one copy P_1 of P can be removed from Q to produce an $(n-1)$-platonic Q'. By induction, the $(n-1)$st repetition of the exposed hinged dissection can fold into Q'. Also, P_1 itself can be decomposed into an instance of the exposed hinged dissection. Our goal is to merge these two hinged dissections.

Let P_2 denote a copy of P in Q' that shares a face f with P_1. Suppose the exposed cyclic hinged dissection of P consists of pieces $H_1, H_2, \ldots, H_k$ in that order. Let H_i denote the piece in the hinged dissection of P_2 incident to face f. Let h be a hinge incident to f (which must be an edge of f) and thus incident to H_i. Suppose by symmetry that the other piece in Q' incident to hinge h is H_{i+1}.

Then we rotate P_1 so that its piece H_{i+1} is flush against the H_i piece in P_2, along the shared face f between P_1 and P_2. We further rotate P_1 so that the hinge h' between pieces H_i and H_{i+1} in P_1 aligns with the hinge h between pieces H_i and H_{i+1} in P_2. We then replace hinges h and h' with two hinges, one from H_i in P_2 to H_{i+1} in P_1, and the other from H_i in P_1 to H_{i+1} in P_2. The resulting hinged dissection is a single cycle, and every instance of piece H_i hinges to pieces H_{i-1} and H_{i+1}, so the resulting hinged dissection is a folding of the nth repetition of $H_1, H_2, \ldots, H_k$ as desired. $\square$

Corollary 1. *If P is the platonic solid with f faces, then there is an $(nf/2)$-piece cyclic hinged dissection that can fold into all n-platonics of type P.*

3 Generalized Interior Dissection

The proof of hinged dissections for polyplatonics consists of two main parts: (1) the construction of an exposed cyclic hinged dissection of a single platonic solid, with the property that at most one piece is incident to each face, and (2) an inductive argument about the nth repetition. In this section we generalize the first part to any polyhedron with a Hamiltonian cycle on its faces. The second part will remain restrictive until future sections.

3.1 Exposed Cyclic Hinged Dissections of Hamiltonian Polyhedra

The exposed cyclic hinged dissection for platonic solids from Section 2.1 essentially exploited that platonic solids, like all convex polyhedra, are "star-shaped". A polyhedron is *star-shaped* if it has at least one point c in its interior from which the line segment to any point on the polyhedron's surface remains interior to the polyhedron. Any star-shaped polyhedron can be carved into face-based pyramids with apexes at c. These pyramids can be hinged together cyclically at the edges of the polyhedron crossed by the Hamiltonian cycle on the faces.

Dissection of a polyhedron into face-based pyramids with a common apex is possible precisely when the polyhedron is star-shaped. However, it is not hard to obtain a dissection of an arbitrary polyhedron into one piece per face, though the pieces are no longer pyramids. One approach is to use the *straight skeleton* [2, 1, 10, 6]. The straight skeleton is normally defined as a particular one-dimensional tree structure contained in a given two-dimensional polygon. For our purposes, the relevant property is that the tree structure subdivides the polygon into exactly one region per polygon edge, and only that region is incident to that polygon edge [2].

The straight skeleton can be generalized to 3D as a decomposition of a given polyhedron into exactly one cell per facet, and only that cell is incident to that facet. We imagine sweeping every facet perpendicularly inwards at the same speed in parallel. Faces change geometry as they are inset by clipping or extending to where they meet adjacent faces. Faces may become disconnected, in which case the sweep continues with each piece, or disappear, in which case the sweep continues without that face. In the end, the entire polyhedron is swept, and the regions swept by individual faces form a partition with the desired property that exactly one region is incident to each facet. Erickson [11] points out that the straight skeleton is no longer well-defined in 3D: there are choices during the offset process that can be resolved multiple ways. However, for our purposes, we just need a single straight skeleton, with an arbitrary decision for each choice, for a suitable decomposition.

As before, the pieces can be hinged together cyclically at the edges of the polyhedron crossed by the Hamiltonian cycle. Thus, for any polyhedron with a Hamiltonian cycle on its n faces, we obtain an n-piece exposed cyclic hinged dissection with the property that each face of the polyhedron is incident to exactly one piece.

3.2 Inductive Hinged Dissection

The second part of the argument is the inductive construction. The key steps here are the two rotations of an added piece P_1. The first rotation ensures that the next piece in the hinging of P_1 (H_{i+1}) is against the piece to which we want to join P_1 (H_i of P_2). The second rotation ensures that the exposed hinges of these two pieces coincide.

These rotations enforce restrictions on what types of polypolyhedra we can build. The first rotation essentially requires that all faces of P "look the same" (in addition to having the same shape): the rotation that brings any face to any other face should result in an identical copy of P (but with faces relabeled). The second rotation requires that all orientations of a face look the same. Unfortunately, these two restrictions force P to be a platonic solid. The goal of the remaining sections is to remove these restrictions, in addition to the restriction that P has a Hamiltonian cycle on its faces.

4 Surface Refinement

In this section we remove two constraints on the base polyhedron P: the requirement that P has a Hamiltonian cycle on its faces, and the requirement that all faces of P look the same. We achieve both of these generalizations by subdividing each face of P by a collection of linear cuts.

First, we divide each reflectionally symmetric face of P along one of its lines of symmetry. Recall that joinings between copies of P are possible only along reflectionally symmetric faces. Now if we can arrange for these symmetry lines to be hinges in an exposed cyclic hinged dissection of the new polyhedron P', then whenever we attempt to attach a new piece P'_1, we are guaranteed that the two consecutive pieces H_i and H_{i+1} of the hinging that we need to place against each other are in fact the two reflectional halves of the original face. Thus the first rotation in the induction construction does exactly what we want: it brings together the two identically labeled faces of P.

Second, we divide each face of P' so that any spanning tree of the faces in P' translates into a Hamiltonian cycle in the resulting polyhedron P''. This reduction is similar to the Hamiltonian triangulation result of [4] as well as a refinement for hinged dissection of 2D polyforms [7, Section 6]. We conceptually triangulate each face f of P' using chords (though we do not cut along the edges of that triangulation). Then, for each triangle, we cut from an arbitrarily chosen interior point to the midpoints of the three edges. Figure 6 shows an example of this process. For any spanning tree of the faces of P', we can walk around the tree (i.e., follow an Eulerian tour) and produce a Hamiltonian cycle on the faces of P''.

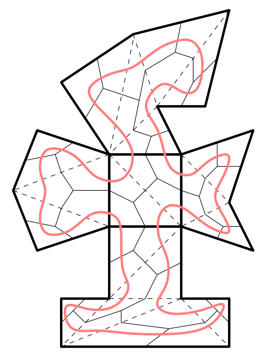

Fig. 6. Hamiltonian refinement of five faces in a hypothetical polyhedron, shown here unfolded. Bold lines outline faces. Dashed lines show triangulations and are not cuts. Thin solid lines are cuts. The curved line shows a Hamiltonian cycle induced by the spanning tree of this unfolding.

In particular, we can start from the matching on the faces of P' from the reflectionally symmetric pairing, and choose a spanning tree on the faces of P' that contains this matching. Then the resulting Hamiltonian cycle in P'' crosses a subdivided edge of every line of symmetry. (In fact, the Hamiltonian cycle crosses every subdivided edge of every line of symmetry.) Thus, in the exposed cyclic hinged dissection of the Hamiltonian polyhedron P'', there is an exposed hinge along every line of symmetry. Therefore all joinings between copies of P'' can use these hinges, which means that the first rotation in the induction construction happens automatically from joining along corresponding faces.

5 Mutually Rotated Base Polyhedra: Twisters

The last generalization concerns the second rotation in the inductive construction. If every reflectionally symmetric face has only one line of symmetry, this second rotation is automatic just from making the faces meet geometrically.

However, if a face has more than one line of symmetry, the polypolyhedron may require different rotations of the two base polyhedra around their common face.

To enable these kinds of joinings, we introduce the *twister gadget* shown in Figure 7. This gadget allows the top face to rotate by any integer multiple of $360°/k$ with respect to the bottom face. The volume occupied by the twister gadget is a prism with a regular k-gon as a base.

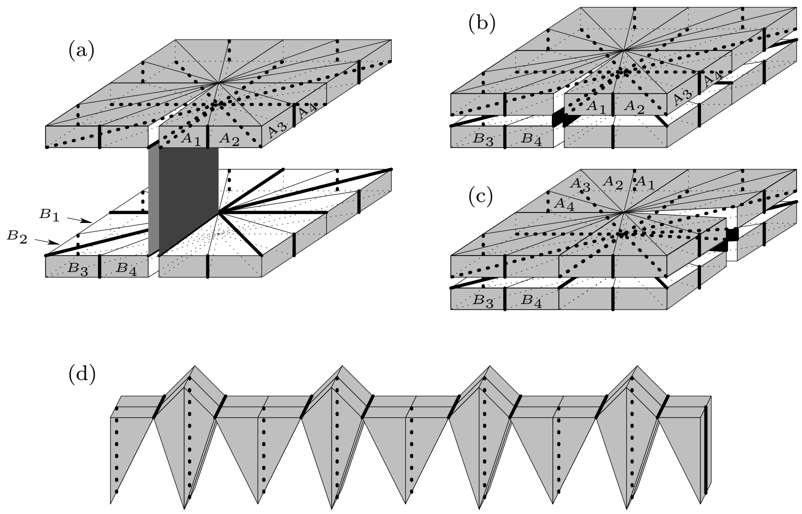

Fig. 7. The twister gadget with $k = 4$: 32 pieces allowing any between none and three quarter turns. For visual clarity, the two layers are drawn substantially separated in (a) and slightly separated in (b) and (c); in fact they are flush. (d) shows the result of unfolding along the perimeter hinges. (c) shows a refolding that achieves a half turn.

To construct the pieces, we slice this prism in half parallel to the base, leaving two identical prisms, one stacked atop the other. Then we divide each prism by making several planar cuts perpendicular to the base: in projection of a regular k-gon, we cut from the center to every vertex, to the midpoint of every edge, and to each quarter point between a vertex and an edge midpoint. The resulting $8k$ pieces are all triangular prisms.

We hinge these prisms together cyclicly as follows. Two hinges connect the top and bottom levels, lying (in projection) along a cut from the center to an edge midpoint. For each remaining cut from the center to an edge midpoint (in projection), and for each cut from the center to a vertex (in projection), there is a hinge connecting the two incident pieces on the "inside" (on the bottom of the top prism and on the top of the bottom prism). For each cut from the center to a quarter point (in projection), there is a hinge connecting the two incident pieces on the perimeter of the regular k-gon.

The perimeter hinges enable the twister to unfold as shown in Figure 7(d) to make all the inside hinges parallel. The inside hinges allow the twister to be further unfolded from this state into a convex three-dimensional "ring". Then we can reverse the process, collapsing the 3D ring back down along the inside hinges to a nearly flat unfolding like Figure 7(d), and folding it back along the perimeter hinges into the regular k-gon configuration. In between the unfolding and folding, by rotating the ring state, we can change which pieces are ultimately on which layer as shown in Figure 7(c).

Specifically, by this continuous folding process, we can move any multiple of 4 pieces from the top layer to the bottom layer on one side of the gap where the layers connect, and the same number of pieces from the bottom layer to the top layer on the other side of the gap. If we move $4j$ pieces on either side, we rotate the top regular k-gon by $j \cdot 360°/k$ relative to the bottom regular k-gon. If we restrict j to satisfy $0 \leq j < k$ (which suffices for the desired set of k possible rotations), then there are four pieces A_1, A_2, A_3, A_4 that always remain on the top layer and four pieces B_1, B_2, B_3, B_4 that always remain on the bottom layer.

To allow the twister gadget to attach to other pieces on its top and bottom, we need to add exposed hinges. We remove the inner hinge connecting A_2 and A_3, which in projection connects the center to a vertex of the regular k-gon, and replace it with a corresponding outer hinge on the top side of the twister gadget. Similarly, we remove the inner hinge connecting B_2 and B_3, whose projection connects the center to the same vertex of the regular k-gon, and replace it with a corresponding outer hinge on the bottom side of the twister gadget. The modified twister gadget can be folded continuously as before, except that now we keep A_2 rigidly attached to A_3 and B_2 rigidly attached to B_3 when opening up into a three-dimensional ring, not folding the two outer hinges at all.

We embed the modified twister gadget in each face of the base polyhedron P that has k-fold symmetry for $k \geq 3$. More precisely, we carve out of P a thin prism with a small regular k-gon base, centered at the symmetry center of the face, and infuse this carved space with a twister gadget. Then we construct the refinement P'' of P as before, choosing an arbitrary line of symmetry of a k-fold symmetric face for the subdivision and resulting matching. The line of symmetry actually now "bends" slightly to dip underneath the thin twister gadget at the center. Normally the hinged dissection of P'' would have a hinge along this line of symmetry, connecting the two incident pieces C and D. Instead, we rotate the embedded twister gadget so that its outer hinges (those between A_2 and A_3 and between B_2 and B_3) align with this chosen line of symmetry, and so that B_2 is atop C and B_3 is atop D. Then we replace the outer hinge between B_2 and B_3 with a hinge between B_2 and C and a hinge between B_3 and D. (All three of these hinges lie geometrically along the same line segment in the folded configuration.)

In the inductive construction of an n-polyhedron of type P, we use the outer hinge between pieces A_2 and A_3 to combine two copies of P''' along a k-fold symmetric face, $k \geq 3$. This hinge lies along the chosen line of symmetry, in the middle of the face, and therefore can be aligned between the two copies. Note that the resulting construction has two copies of the twister gadget joined along their top sides, which is redundant because it allows up to two full turns of the faces, but we cannot easily remove this redundancy while having two identical copies of a single hinged dissection.

Two copies of P''' joined along a face of k-fold symmetry can now rotate with respect to each other by $j \cdot 360°/k$, for any desired $0 \leq j < k$. This property is exactly what we need to perform the second rotation in the inductive argument of hinged dissectibility.

This completes our construction of a hinged dissection that folds into all n-polyhedra of type P, for any positive integer n and for any polyhedron P.

6 Self-Similar Hinged Dissections

This section considers a related side problem from the main line of the paper, called "self-similar hinged dissections". A hinged dissection is *self-similar* if every piece is similar to (a scaled copy of) the base polyhedron P. Self-similar dissections (without hingings) are well-studied in recreational mathematics, usually in 2D, so it is natural to consider their hinged, 3D counterparts.

Figure 8 gives a self-similar exposed hinged dissection of a cube, which by our techniques leads to a self-similar hinged dissection of all n-cubes, for any n. The dissection is simple, dividing the cube into a $2 \times 2 \times 2$ array of identical subcubes. The hinging is less trivial because of the requirement that every face of the original cube has an exposed hinge. The hinges are always between the midpoint of an original edge to the center of an original face, so two hinges between adjacent cubes can always be brought into alignment, after possible rotation around the shared face, during the merging process in the inductive construction.

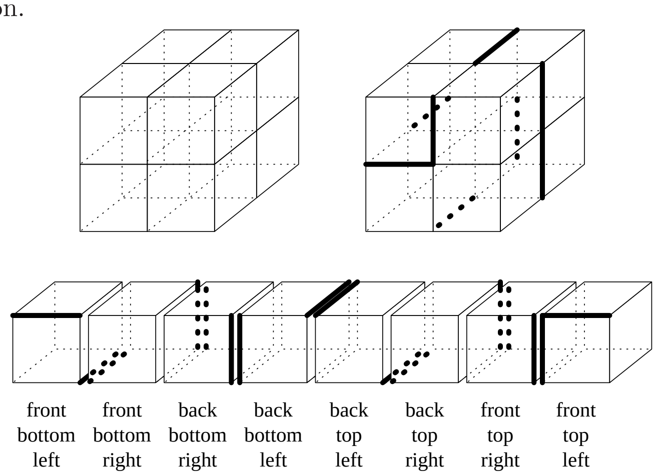

Fig. 8. A hinged dissection of a cube into a $2 \times 2 \times 2$ array of 8 subcubes. This hinged dissection can be used in place of that in Figure 5; every face has (at least) one exposed hinge. Top-left: The dissection. Top-right: The cyclic hinging. Bottom: Unfolded after cutting one hinge. Hinges are drawn bold.

The resulting dissection of n-cubes uses $8n$ pieces (compared to $3n$ pieces from Corollary 1):

Theorem 3. *The nth repetition of the cyclic hinged dissection in Figure 8 consists of $8n$ identical cubes and folds into all n-cubes.*

This hinged dissection of a cube is clearly the smallest exposed self-similar hinged dissection of the cube, and hence is optimal among such dissections. The hinged dissection also applies more generally to any parallelepiped (e.g., an $x \times y \times z$ box) as the base shape P.

This extension has been used in an interactive sculpture [18] consisting of roughly a thousand identical wooden blocks (boxes) hinged together according to Figure 8. (For manipulation purposes, the chain was broken into small segments.)

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