Short Interlocked Linkages

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Abstract

We study collections of $O(1)$ linkages in 3-space, each with $O(1)$ joints, that are *interlocked* in the sense that, without one link crossing through another, they cannot be separated from one another. Our main results are proofs that a triangle and an open 4-chain can interlock, as can a quadrilateral and an open 3-chain. For two 3-chains, we establish that an open 3-chain cannot interlock with a triangle but it can interlock with an open rigid 3-chain, and that two "revolute" 3-chains can interlock. Finally we make several conjectures on which pairs of short, open chains can interlock.

1 Introduction

Consider a simple polygonal chain embedded in 3-space, either an open *arc* or a closed *polygon*. We view the vertices of the chain (except the endpoints of an open chain) as universal *joints*, and the edges of the chain as rigid *bars* (or *links*). A *motion* of the chain is a motion of the vertices that preserves the length of the bars, and never crosses any bars.[1] In particular, a *straightening* of an open chain is a motion after which all angles are 180°. We say that a collection of disjoint, simple chains can be *separated* if each chain can be moved arbitrarily far from each other chain, with all motions avoiding crossings. If a collection cannot be separated, we say its chains are *interlocked*; a single chain that cannot be straightened is called *locked*.

It is known that even single open chains of as few as 5 links in 3-space can be locked [CJ98, BDD+99]. Various classes of chains are known to be unlocked, but it remains unsolved how to decide whether a given chain can be unlocked.

Our work is inspired by a question posed by Anna Lubiw [DO00]: into how many pieces must a chain be cut (say, at vertices) so that the pieces can be separated and straightened? This problem is motivated by proteins, which can be modeled by polygonal chains, and, according to some theories, temporarily split apart in order to achieve the minimum-energy folding. Two easy bounds on Lubiw's problem are that some n-link chains require removal of at least $\lfloor (n-1)/4 \rfloor$ vertices for disentanglement, and no chain needs removal of more than $\lfloor (n-1)/2 \rfloor$ vertices. The lower bound is obtained by concatenating many copies of the 5-link "knitting needles" example from [CJ98, BDD+99], each sharing one link with the next, and observing that each copy of the locked 5-link chain must have one of its four interior vertices cut. See Fig. 1. For the upper bound, the first

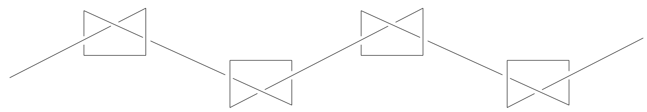

Figure 1: An $n = 17$ link chain requires removal of at least $\lfloor (n-1)/4 \rfloor = 4$ vertices to disentangle.

author and Ming-wei Wang observed that cutting every second joint suffices to disentangle the chain. For, as Jack Snoeyink[2] pointed out, the resulting 2-bar pieces ("hairpins") can be rigidly separated arbitrarily far by blowing up from a point, because the pieces are star-shaped sets [Tou85, Daw84]. It remains open whether removal of every third vertex suffices to disentangle an arbitrary chain, which would imply an upper bound of $\lceil (n-1)/3 \rceil$ on Lubiw's problem.

A natural approach to Lubiw's problem is to find two short interlocked chains, and then construct an n-link chain from this by repeated doubling. We can show that if there exists such an interlocked pair of open chains of k_1 and k_2 links, then a lower bound of $\lfloor (n-1)/(k_1+k_2) \rfloor$ is established. Unfortunately, because either k_1 or k_2 is at least 3, this is weaker than the known $n/4$ bound.

While our original interest in interlocked open chains was motivated by connections to Lubiw's problem, we explore closely related problems of interlocked closed and open chains. In addition we introduce several increasingly constrained versions of open chains: *flexible* chains, those with universal joints (the default); *revolute* chains, those whose joints permit only revolute motion, maintaining a fixed angle between the incident links, and *rigid* chains, those whose joints are frozen. Revolute chains are inspired by protein molecules, which maintain a fixed bond angle between adjacent residues. Their study in computational geometry was initiated in [ST00]. We establish several results, and conjecture a few more, as summarized in Table 1.

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[1]Bars are allowed to touch, and even slide along each other, so long as they do not properly cross.

[2]Personal communication to first author, 2000.

Chain 1		Chain 2		
type	n_1	*type*	n_2	*Interlock?*
open	2	open	2	Proof: NO
closed	3	open	4	Proof: YES
closed	4	open	3	Proof: YES
closed	3	open	3	Proof: NO
open rigid	3	open	3	Claim: YES
open revolute	3	open revolute	3	Claim: YES
open	3	open	3	Conj: NO
open	4	open	4	Conj: YES
open	3	open	5	Conj: YES

Table 1: Proofs, Claims, and Conjectures. Chains are flexible unless otherwise specified. (Claims are not proven in this abstract.)

2 Triangle and 4-chain

Theorem 1 *A triangle and a $(k \geq 4)$-chain can interlock.*
Proof: The configuration is the following: the triangle abc lies in the plane H, the center of its circumcircle is o, and the radius of the circumcircle is r. Call H^+ (resp. H^-) the halfspace above (below) H.

The 4-chain alternates points and links $p_0, l_0, p_1, l_1, ..., l_3, p_4$. p_0 is placed in H^-, l_0 crosses the interior of $\triangle abc$, and ends at p_1, above o. l_1 crosses the interior of $\triangle abc$ again and $p_2 \in H^-$, l_2 crosses H outside of $\triangle abc$, and l_3 crosses the wedge formed by l_0 and l_1 above H. So $\{p_0, p_2\} \subset H^-$ and $\{p_1, p_3, p_4\} \subset H^+$. See Fig. 2.

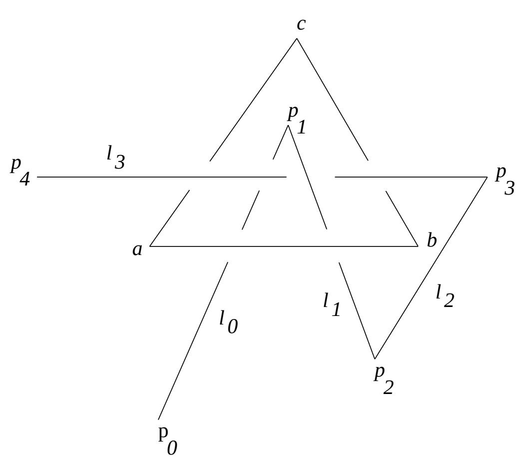

Figure 2: A triangle and a 4-chain can lock.

Let $R = r + |l_1| + |l_2|$, and set the length of l_0 and l_3 to $20R$. Consider the open ball B of radius $15R$, and the ball B' of radius $4R$, both centered at o. Initially, p_0 and p_4 lie outside of B, while a, b, c, p_1, p_2 and p_3 all lie inside $B' \subset B$. As long as p_0 and p_4 stay outside B, we can attach a sufficiently long unknotted string between p_0 and p_4 which remains outside B, and our configuration is equivalent to the link known[3] as 5^2_1. The unlocked configuration corresponds to two separable unknots, so any motion unlocking this configuration would require p_0 or p_4 to enter the ball B. Before that happens, p_0 or p_4 will have to enter the ball B.

Consider the first event when p_0 or p_4 touches the boundary of B. At that point, p_1, p_2 and p_3 must be out of B' but still inside B; see Fig. 3. Thus the only

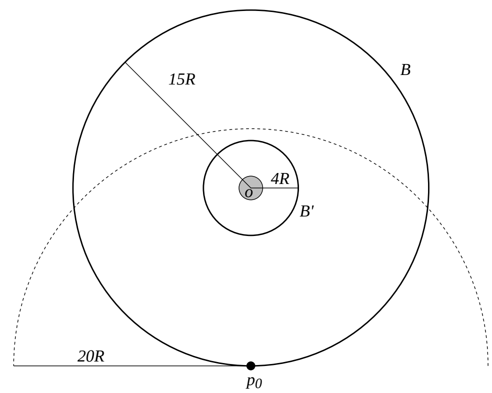

Figure 3: When p_0 touches B, p_1 must be exterior to B' by at least R, and therefore p_2 and p_3 are also exterior to B'.

elements possibly inside B' are $\triangle abc$ and the two links l_0 and l_3. Then either one of l_0 and l_3 cross the interior of $\triangle abc$, or both do, or neither do. The first case corresponds to a link 2^2_1, and the last case to two separable unknots (0^2_1). The second case is more interesting and requires a more careful analysis. Because p_0 and p_4 are still outside of B (B is an open ball), we can replace the string joining them by a great arc γ on the boundary of B. Let T be the plane parallel to l_0 and l_3, and passing through o. Consider the orthogonal projection of our linkage onto T. Note that in the projection, the lengths of l_0 and l_3 are preserved, and all other segment lengths are at most their original lengths. Then, p_1, p_2 and p_3 lie in a ball of radius R centered at p_1, and since they are outside B', we can separate the projections of p_1, p_2 and p_3 from the projections of p_0, p_4 and $\triangle abc$ by a line (this separation is necessary to exclude cases such as the one shown in Fig. 4). From this, we can infer that this configuration is equivalent to two separated unknots (0^2_1), or to 4^2_1 (Fig. 5), neither of which are equivalent to our starting configuration; so this first event could never happen. $\square$

3 Quadrilateral and 3-chain

Theorem 2 *A 4-gon can interlock with a 3-chain.*

[3] Tables of links are often organized by (minimum) crossing number. Thus 5^2_1 represents a link that can be drawn with 5 crossings. Six crossings are used in Fig. 2, but the $l_0 : l_3$ crossing could be removed. The superscript in the link notation is the number of components, for us always 2. (The subscript is an arbitrary table index.) See [Ada94, p. 287] or [Wei99, p. 1086].

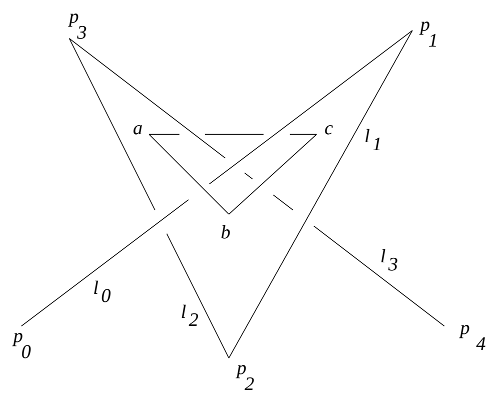

Figure 4: This configuration is incompatible with the fact that p_0 or p_4 touch the boundary of B.

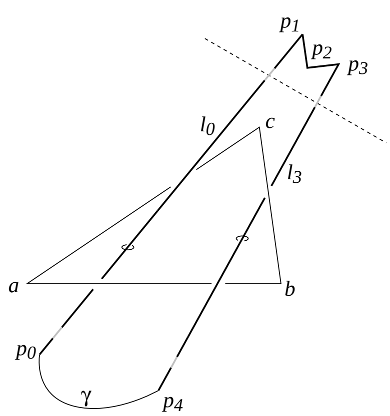

Figure 5: The link 4_1^2, formed when l_0 and l_3 both pass through the interior of $\triangle abc$. (Not to scale; gray segments indicate omissions.) $\{p_1, p_2, p_3\}$ can be separated from $\{a, b, c, p_0, p_4\}$.

Proof: Starting with the configuration of Fig. 6, let $R = |ab| + |bc| + |cd| + |l_1|$ and set the length of l_0 and l_2 to $20R$. Consider the open ball B of radius $15R$, the ball B' of radius $4R$, both centered at a. As in the previous proof, we connect p_0 to p_3 by a string staying out of B. The resulting link is now 7_1^2 (in a variant of this configuration, l_0 is above l_2, and the link is 6_1^2). We again argue that in order to separate the 4-gon from the 3-chain, p_0 or p_3 has to enter the ball B. The first time one of those endpoints touch the boundary of B, p_1 and p_2 are still inside B but out of B' and the only elements possibly inside B' are $abcd$ and the two edges l_0 and l_2.

We now argue that the possible cases are the same as in the previous theorem. Consider the tetrahedron formed by the vertices $abcd$, assign a weight of $+1$ to the faces abd and bcd, and a weight of -1 to abc and acd. The case of an edge not intersecting the tetrahedron and the case of an edge crossing the tetrahedron at two faces of same sign are equivalent to the case of the

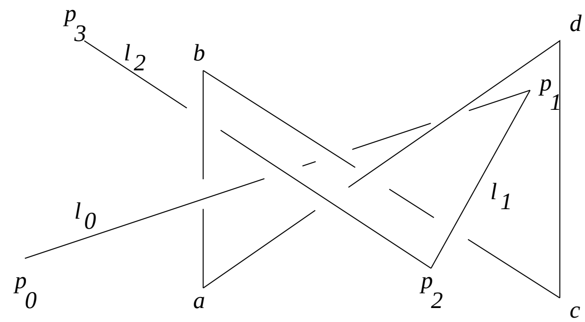

Figure 6: A quadrilateral and a 3-chain can lock.

edge not intersecting the interior of the triangle in the previous theorem, and the case of an edge intersecting faces of opposite sign is equivalent to the case of an edge intersecting the interior of the triangle. The rest of the proof is identical. □

4 Two 3-chains

In this section we explore a variety of interactions between two 3-chains.

We take it as obvious that a 2-chain may always be separated from a triangle. To show that Theorem 1 is sharp, we now prove that a 3-chain may always be separated from a triangle.

Define a *T-chain* to be an edge e_1 attached at one end to a point p along another edge e_2. A valid motion for a T-chain preserves edge lengths and the distance between the connection point p and the endpoints of e_2. Note that a 2-chain is a T-chain.

Lemma 3 *A T-chain can always be separated from a triangle by a translation.*

Theorem 4 *An open 3-chain cannot interlock with a triangle.*
Proof: (Sketch.) We follow the notation established in Theorem 1: $\triangle abc$ lies in plane H, and the 3-chain C has vertices (p_0, p_1, p_2, p_3) and links (l_0, l_1, l_2). First assume C is not planar. Otherwise, make C nonplanar by a small motion. Let $q_i = l_i \cap H$.

The proof considers several cases; for all but one we can move C to be similar to a T-chain configuration. Here we only examine the remaining case: both l_0 and l_2 intersect $\triangle abc$ but l_1 does not, with $\{p_0, p_2\} \in H^+$ and $\{p_1, p_3\} \subset H^-$.

Because $\{q_0, p_1, p_2, q_2\}$ do not lie on a common plane, or equivalently, $\{q_0, q_1, q_2\}$ are not collinear, C is non-self-intersecting; see Fig. 7. In fact, for any position of l_1 such that $(l_1 \cap H) \notin L_{0,2}$, the lines containing q_0p_1 and p_2q_2 do not intersect, and do not intersect the edges of $\triangle abc$. Thus the motion that translates l_1 in a direction orthogonal to $L_{0,2}$ and parallel to H, away from $\triangle abc$, while maintaining l_0 and l_2 through the

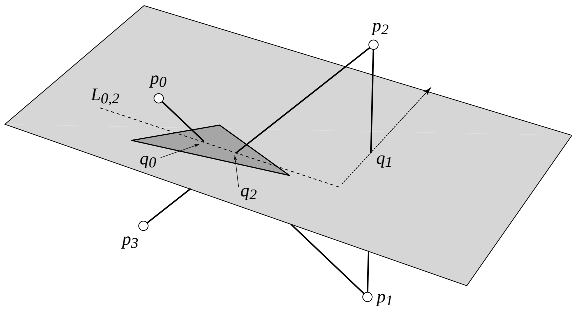

Figure 7: Translating l_1 so that $q_1 = l_1 \cap H$ moves away from $L_{0,2}$ separates the 3-chain from $\triangle abc$.

original q_0 and q_2 points, will avoid self-intersection, and eventually pull a link out of $\triangle abc$. Then we fall into a previous case. □

In light of Theorem 4, the following is surprising:

Claim 5 *An open, rigid 3-chain can interlock with an open, flexible 3-chain.*

One example is shown in Fig. 8.[4] The rigid 3-chain has vertices at $r_0 = (1, 5, 0)$, $r_1 = (1, 0, 0)$, $r_2 = (0, 1, 1)$, $r_3 = (5, 1, 0)$. The flexible 3-chain has interior vertices at $p_1 = (1+\epsilon, 1+\epsilon, 1+2\epsilon)$ and $p_2 = (1+\epsilon, 1+\epsilon, -2\epsilon)$, for small $\epsilon > 0$, and $p_0 = (-2\epsilon, 0, -4\epsilon)$ and $p_3 = (0, -2\epsilon, 1+4\epsilon)$. Attempting, for example, to translate l_1 of the flexible chain in the $(1, 1, 0)$-direction causes the crossed links l_0 and l_3 to "scissor" the central rigid link $r_1 r_2$.

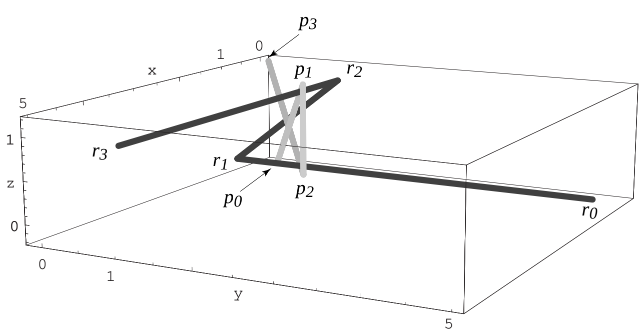

Figure 8: A rigid 3-chain (dark) interlocked with a flexible 3-chain (light).

Our final result, which we have no room to detail here, is the following:

Claim 6 *Two revolute 3-chains can interlock.*

5 Discussion

We believe that the closed 3-chain in Fig. 2 can be replaced by an open 4-chain, resulting in two interlocked open 4-chains. Similarly, we believe that the closed 4-chain in Fig. 6 can be replaced by an open 5-chain, resulting in an interlocked pair of open 3- and 5-chains. This motivates two of the three conjectures in Table 1.

The third conjecture is that two open 3-chains cannot interlock, which would complete our exploration of two 3-chains in Section 4.

Finally, even settling these three conjectures leaves an interesting case: Can an open 3-chain lock with an open 4-chain? See Fig. 9.

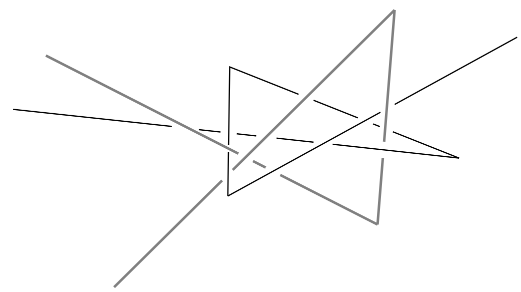

Figure 9: An open 3-chain (shaded) tangled with an open 4-chain (black): Are they interlocked?

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[4] See `http://db.uwaterloo.ca/~eddemain/interlocked/` for more effective three-dimensional images.