Folding and Unfolding Linkages, Paper, and Polyhedra

Erik D. Demaine

Department of Computer Science, University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada, eddemaine@uwaterloo.ca

1 Introduction

Folding and unfolding problems have been implicit since Albrecht Dürer in the early 1500's [Dür77], but have not been studied extensively until recently. Over the past few years, there has been a surge of interest in these problems in discrete and computational geometry. This paper gives a brief survey of some of the recent work in this area, subdivided into three sections based on the type of object being folded: linkages, paper, or polyhedra. See also [O'R98] for a related survey from this conference two years ago.

In general, we are interested in how objects (such as linkages, paper, and polyhedra) can be moved or reconfigured (folded) subject to certain constraints depending on the type of object and the problem of interest. Typically the process of *unfolding* approaches a more basic shape, whereas *folding* complicates the shape. We can also generally define the *configuration space* as the set of all configurations or states of the object, with paths in the space corresponding to motions (foldings) of the object.

2 Linkages

A *linkage* or *framework* consists of a collection of rigid line segments (*bars*) joined at their endpoints (*vertices*) to form a particular graph. A linkage can be *folded* by moving the vertices around in $\mathbb{R}^d$ in any way that preserves the length of each bar. Such linkages have been studied extensively in the case that bars are allowed to cross; see, for example, [KM95,LW95,Sal73,Whi92]. Recently there has been much work on the case that the linkage must remain *simple*, never crossing any two bars.[1] This additional constraint is the type of linkage folding considered in this section. Such linkage folding has applications in hydraulic tube bending [O'R98] and motion planning of robot arms. There are also connections to protein folding in molecular biology. See [CDR00,O'R98,Tou99a] for other surveys on this area.

Perhaps the most fundamental question we can ask about folding linkages is whether it is possible to fold between any two simple configurations of the

[1] Typically, bars are allowed to touch, provided they do not properly intersect. However, requiring bars to touch only at common endpoints does not change the results.

same linkage (with matching graphs, combinatorial embeddings, and bar lengths) while preserving the bar lengths and not crossing any bars during the folding. Because folding motions can be reversed and concatenated, this fundamental question is equivalent to whether every simple configuration can be folded into some *canonical configuration*.

In this context, three general types of linkages are commonly studied, characterized by the structure of their associated graphs: a *polygonal arc* or *open polygonal chain* (a single path); a *polygonal cycle*, *polygon*, or *closed polygonal chain* (a single cycle); and a *polygonal tree* (a single tree). The canonical configuration of an arc is the *straight configuration*, all vertex angles equal to 180°. A canonical configuration of a cycle is a *convex configuration*, planar and having all interior vertex angles less than or equal to 180°. It is relatively easy to show that convex configurations are indeed "canonical" in the sense that any one can be folded into any other; this result was first proved in the literature in [ADE+01]. Finally, a canonical configuration of a tree is a *flat configuration*: all vertices lie on a horizontal line, and all bars point "rightward" from a common root. Again it is easy to fold any flat configuration into any other.

The fundamental questions thus become whether every arc can be straightened, every cycle can be convexified, and every tree can be flattened. The answers to these questions depend on the dimension of the space. Over the past few years, this collection of questions has been completely resolved. A summary is shown in Table 1. In the remainder of this section, we describe the historical progress of these results, and describe other results on linkage folding not captured by this categorization.

Dimension	Can all arcs be straightened?	Can all cycles be convexified?	Can all trees be flattened?
2	Yes [CDR00]	Yes [CDR00]	No [BDD+01]
3	No [CJ98,BDD+99]	No [CJ98,BDD+99]	No
4 & above	Yes [CO99]	Yes [CO99]	Yes

Table 1. Answers to main linkage-folding problems.

The questions of whether every polygonal arc can be straightened in the plane and whether every polygon can be convexified in the plane have arisen in many contexts over the last quarter-century. In particular, they were posed independently by Stephen Schanuel and George Bergman in the early 1970's, Ulf Grenander in 1987, William Lenhart and Sue Whitesides in 1991, and Joseph Mitchell in 1992. In the discrete and computational geometry community, the arc-straightening problem has become known as the *carpenter's rule problem* because a carpenter's rule folds like a polygonal arc.

Many people devoted time to these two problems over the past 10 years. It was widely conjectured, particularly by those unfamiliar with the problem, that the answers were yes. On the other hand, several people proposed examples of

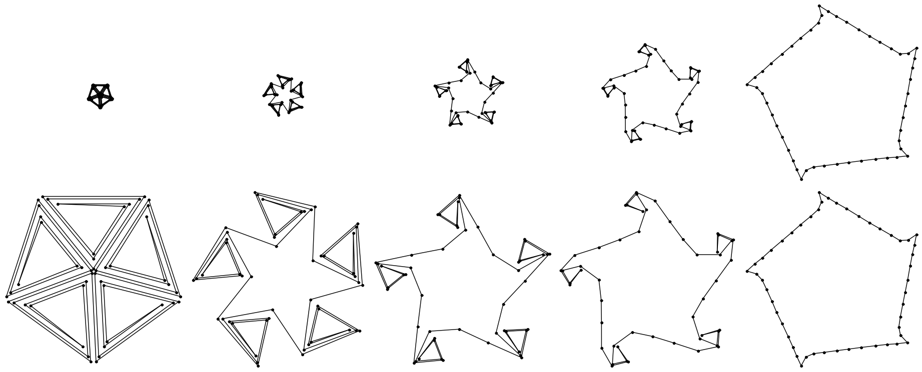

Fig. 1. Two views of convexifying a "doubled tree." The top snapshots are all scaled the same, and the bottom snapshots are scaled differently to improve visibility. See `http://daisy.uwaterloo.ca/~eddemain/linkage/` for more animations.

polygonal arcs and cycles that might be "locked" (unstraightenable and unconvexifiable), but eventually every proposed example was unlocked by hand. It was not until early in the year 2000 that the problems were solved in the positive by Connelly, Demaine, and Rote [CDR00]. See [CDR00] for a more detailed history.

More generally, the result in [CDR00] shows that a collection of nonintersecting polygonal arcs and cycles in the plane may be simultaneously folded so that the outermost arcs are straightened and the outermost cycles are convexified. The "outermost" proviso is necessary because arcs and cycles cannot always be straightened and convexified when they are contained in other cycles. The key idea for the solution, introduced by Günter Rote, is to look for *expansive* motions in which no vertex-to-vertex distance decreases. Expansive motions automatically preserve simplicity, so the difficult noncrossing aspect of the problem can be ignored by guaranteeing expansiveness. This idea allowed applying theorems in rigidity theory and tensegrity theory to show that, *infinitesimally*, arcs and cycles can be folded expansively. These infinitesimal motions are combined by flowing along a vector field implicitly defined by an optimization problem. As a result, the motion is piecewise-differentiable, and the configuration space of arcs and cycles is contractible. In addition, any symmetries present in the initial configuration of the linkage are preserved throughout the motion. Similar techniques show that the area of each cycle increases by this motion and furthermore by any expansive motion [CDR00].

Ileana Streinu [Str00] has demonstrated another motion for straightening arcs and convexifying polygons that is piecewise-algebraic, made up of $O(n^2)$ mechanisms each with one degree of freedom. As a result, the motion is possible to compute in principle. On the other hand, an approximation to the motion in [CDR00] is easy to implement, and has resulted in animations such as the one in Fig. 1.

The pursuit of the arc-and-cycle problems in 2D inspired research on several related problems. For example, it was shown that starshaped polygons [ELR+98] and monotone polygons [BDL+99] can be convexified by particularly simple motions. Biedl et al. [BDD+01] showed that a positive answer to the arc-and-cycle problem could not be generalized to flattening trees. Recently, Connelly, Demaine, and Rote have shown that even a tree with one degree-3 vertex and the remaining vertices degree-2 can be locked (manuscript in preparation, October 2000), so the result in [CDR00] is tight.

Linkage folding in 3D was initiated earlier, by Paul Erdős in 1935 [Erd35]. He asked whether a particular "flipping" algorithm for folding a planar polygon through three dimensions (preserving edge lengths and simplicity) would convexify the polygon in a finite number of steps. With a slight modification, this question was answered positively by Nagy [Nag39]. This problem and result have been rediscovered several times; see [Tou99b,Grü95] for the history. Unfortunately, Erdős's algorithm (or more precisely, Nagy's modification) can require arbitrarily many moves, even for a quadrangle [Grü95,Tou99b,BDD+99]. Recently, algorithms that convexify planar polygons through 3D in a linear number of "simple moves" have been developed [BDD+99,AGP99]. More generally, if a polygonal arc or cycle in 3D has a simple orthogonal projection, then it can be straightened or convexified [CKM+01]; interestingly, this result is based on the 2D result [CDR00]. But if we start with a general polygonal arc or an unknotted polygon in 3D, it is not always possible to straighten or convexify it [CJ98,Tou01,BDD+99]; see Fig. 2 for an example of a locked arc in 3D. Other problems related to Erdős flips include flipturns, described elsewhere in this proceedings [ACD+00,ABC+00,Grü95], and deflations [FHM+01,Grü95].

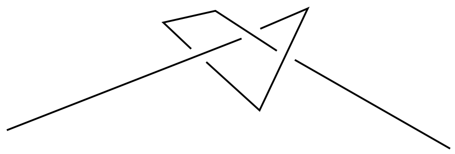

Fig. 2. A locked polygonal arc in 3D with 5 bars [CJ98,BDD+99].

Finally, analogous to the nonexistence of knots in dimensions higher than 3, polygonal arcs can be straightened and polygonal cycles can be convexified in 4D and higher dimensions [CO99]. Intuitively, this result holds because the number of degrees of freedom of any vertex is much higher than the dimensionality of the obstacles imposed by any bar. It would be interesting to explore scaling the dimension of the object to be folded together with the dimension of the space in which it is folded. For example, how can solid polygons connected at their edges be folded in dimensions higher than 2?

3 Paper

Paper folding (origami) has lead to several interesting mathematical and computational questions over the past fifteen years or so. A piece of paper, normally a (solid) polygon such as a square or rectangle, can be folded by any continuous motion that preserves the distances on the surface and does not properly self-intersect. Informally, paper cannot tear, stretch, or cross itself, but may otherwise bend freely. Formally, a folding is a continuum of isometric embeddings of the piece of paper in $\mathbb{R}^3$. However, the use of the term "embedding" is weak: paper is permitted to touch itself provided it does not properly cross itself. In particular, a *flat folding* folds the piece of paper back into the plane, and so the paper must necessarily touch itself. We frequently identify the continuous motion of a folding with the final folded state of the paper; in the case of a flat folding, the flat folded state is called a *flat origami*.

Some of the pioneering work in origami mathematics [Hul94,Jus94,Kaw89] studies the *crease pattern* that results from unfolding a flat origami, that is, the graph of edges on the paper that fold to edges of a flat origami. Stated in reverse, what crease patterns have flat foldings? Necessary and sufficient conditions are known [Hul94,Jus94,Kaw89], but there is little hope for a polynomial characterization: Bern and Hayes [BH96] have shown that this decision problem is NP-hard. The key difficulty is the non-self-intersection property, more precisely, in finding an overlap order of faces that avoids self-intersection in the folded state. If such nonlocal interactions are ignored, the existence of a flat origami can be tested in linear time [BH96].

A more recent trend, as in [BH96], is to explore *computational origami*, the algorithmic aspects of paper folding. This aspect was pioneered by Lang [Lan96], who has shown how to design a wide class of origami "bases" from which real origami models are folded. In the past two years, computational geometry techniques have been applied to computational origami; we briefly survey these results in the remainder of this section. See also [DD01].

One result involves the *fold-and-cut problem*: given a sheet of paper, fold it flat, make one complete straight cut, and unfold the pieces. What shapes can be achieved? Surprisingly, we can arrange the folds and the cut in order to make any desired plane graph (planar graph embedded with straight edges). See Fig. 3 for some examples. Two solutions to this problem have been developed. Demaine, Demaine, and Lubiw [DDL98] presented a solution based on the straight skeleton at this conference two years ago. Bern, Demaine, Eppstein, and Hayes [BDEH98] developed a different solution based on disk packing.

Another surprising "universality" result in paper folding is about folding silhouettes and wrapping polyhedra. Given a polygon in the plane, possibly with holes, can we fold a sufficiently large piece of paper into that silhouette? This question is implicit throughout origami design, and was first formally stated in [BH96]. If the paper has a different color on each side, and the polygon is partitioned into differently colored regions (as in Fig. 4), can we fold the paper into that shape with the appropriate colors showing at the appropriate regions? More generally, if the desired shape is not a flat silhouette but a general con-

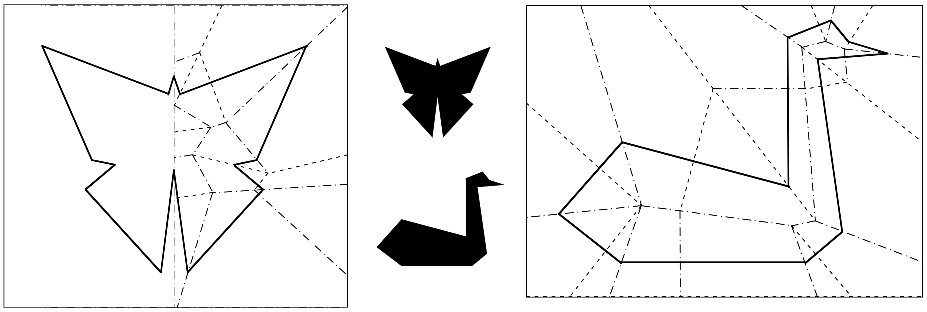

Fig. 3. Crease patterns for folding a rectangle of paper flat so that one complete straight cut makes a butterfly (left) or a swan (right). Valley creases are drawn with dotted lines, and mountain creases are drawn dash-dotted.

nected union of polygons in 3-space (a "polyhedron"), can such a package always be tightly wrapped by a sufficiently large piece of paper, possibly matching a 2-color pattern? Demaine, Demaine, and Mitchell [DDM00] have shown that the answers to all of these questions are yes, and describe three algorithms for solving these problems. Several problems concerning the efficiency of the foldings remain open.

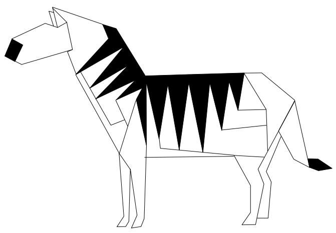

Fig. 4. A flat folding of a square of paper, black on one side and white on the other side, designed by John Montroll [Mon91, pp. 94–103].

Returning to the problem of recognizing flat-foldable crease patterns, an interesting special case is *map folding*. More precisely, a *map* is a rectangle with horizontal and vertical creases, each marked either mountain or valley. While map folding is normally only studied from the combinatorial perspective [Gar83, Lun71], Jack Edmonds (personal communication, August 1997) posed two attractive decision questions: (1) does a given map have a flat folded state, and (2) can a given map be folded flat by a sequence of simple folds (each folding along one line)? The complexity of the first problem remains open; an NP-hardness result would be an interesting strengthening of [BH96]. Recently, Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, and Skiena [ABD$^+$00] resolved the complexity of the second problem. The exact results depend on the model of simple folds: if the paper can be folded one layer at a time, then foldability can be decided in linear time; if all layers must be folded at once (a more restrictive model), then foldability can be decided in near-linear time, e.g., $O(n \log n)$. Surprisingly, however, map folding is on the border of computational intractability: the same question with folds allowed at 45 degrees, or with a nonrectangular piece of paper, is (weakly) NP-complete [ABD+00].

4 Polyhedra

Unlike the other problems, there are several different models of folding that arise in the context of polyhedra.

A classic open problem is whether (the surface of) every convex polyhedron can be cut along some of its edges and unfolded into the plane without overlap [She75,O'R98]. Such unfoldings go back to Dürer [Dür77], and have important practical applications in manufacturing, such as sheet-metal bending. It is widely conjectured that the answer to this question is yes, but all attempts at a solution have so far failed. Experiments by Schevon [Sch89,O'R98] suggest that a random unfolding of a random polytope overlaps with probability 1, but this does not preclude the existence of at least one nonoverlapping unfolding for all polyhedra.

Instead of answering this difficult question directly, we can examine to what extent it can be generalized. In particular, define a polyhedron to be *topologically convex* if its 1-skeleton (graph) is the 1-skeleton of a convex polyhedron. Does every topologically convex polyhedron have such an edge unfolding? Bern, Demaine, Eppstein, Kuo, Mantler, and Snoeyink [BDE+01] have shown that the answer is no: there is a polyhedron homeomorphic to a sphere and with every face a triangle that has no (one-piece, nonoverlapping) edge unfolding. It is shown in Fig. 5. The complexity of deciding whether a given topologically convex polyhedron can be edge-unfolded remains open.

Another intriguing open problem in this area is whether every polyhedron homeomorphic to a sphere has *some* one-piece unfolding, not necessarily using cuts along edges. It is known that every convex polyhedron has an unfolding in this model, allowing cuts across the faces of the polytope [AO92,MMP87]. But many nonconvex polyhedra also have such unfoldings. For example, Fig. 5 illustrates one for the polyhedron described above. Biedl, Demaine, Demaine, Lubiw, Overmars, O'Rourke, Robbins, and Whitesides [BDD+98] have shown how to unfold many orthogonal polyhedra, even with holes and knotted topology, although it remains open whether all orthogonal polyhedra can be unfolded. The only known scenario that prevents unfolding altogether [BDE+01] is a polyhedron with a single vertex of negative curvature (more than 360° of material), but this requires the polyhedron to have boundary, edges incident to only one face.

In addition to unfolding polyhedra into simple planar polygons, we can consider the reverse problem of folding polygons into polyhedra. Lubiw and O'Rourke [LO96] have shown how to test in polynomial time whether a polygon

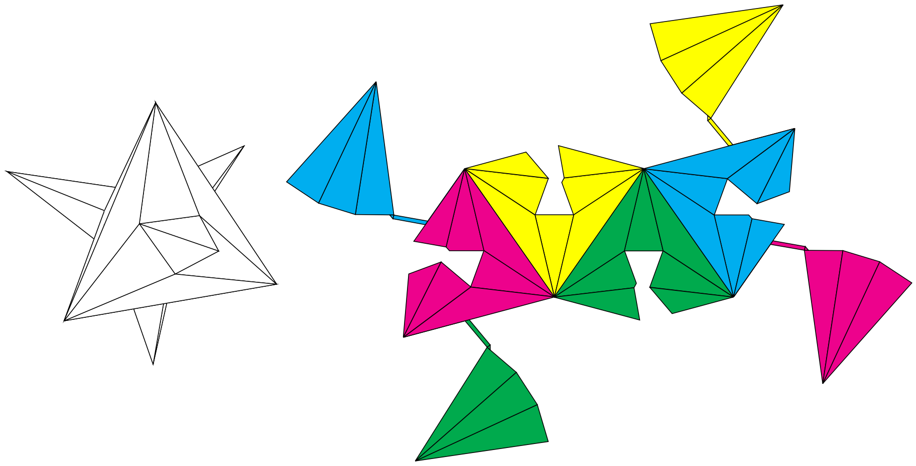

Fig. 5. (Left) Simplicial polyhedron with no edge unfolding. (Right) An unfolding when cuts are allowed across faces.

has an edge-to-edge gluing that can be folded into a convex polyhedron, and how to list all such edge-to-edge gluings in exponential time. The exponential time is necessary because some examples have that many gluings, as described elsewhere in this proceedings [DDLO00a,DDLO00b]. This work shows several other enumerative and structural results about foldings and unfoldings. We are also working on efficient algorithms for detecting the existence of and enumerating non-edge-to-edge gluings, generalizing [LO96]. An intriguing open problem remains relatively unexplored: a theorem of Aleksandrov implies that any gluing found can be folded into a unique convex polyhedron, but how efficiently can this polyhedron be constructed?

A different kind of polyhedron folding comes from extending the fold-and-cut problem from the previous section to one higher dimension. Given any polyhedral complex, can $\mathbb{R}^3$ be folded (through $\mathbb{R}^4$) "flat" into $\mathbb{R}^3$ so that the surface of the polyhedral complex maps to a common plane, and nothing else maps to that plane? While the applicability of four dimensions is difficult to imagine, the problem's restriction to the surface of the complex is quite practical, e.g. in packing: *flatten* the surface of a polyhedron into a flat folded state, without cutting or stretching the paper. Demaine, Demaine, and Lubiw [DDL00] have shown that convex polyhedra and orthogonal polyhedra can be flattened, among other classes. An example is shown in Fig. 6. We conjecture further that every polyhedral complex can be flattened.

5 Conclusion

The area of folding and unfolding offers many beautiful mathematical and computational problems. Much progress has been made recently in the many problems outlined above, and many more important problems remain open. For ex-

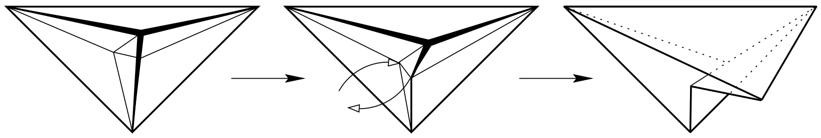

Fig. 6. Flattening a tetrahedron, from left to right. Note that the faces are not flat in the middle picture.

ample, most aspects of unfolding polyhedra remain unsolved, including the original problem in the area, edge-unfolding convex polyhedra. A variety of results suggest that paper folding possesses a vast power, but what is known is certainly not the whole story of what is possible. And while the described class of linkage problems has been resolved, there are several other aspects that remain unstudied. For example, protein folding is a domain of great practical importance in biology that should be the source of many interesting geometric problems, with connections to linkages. But even more exciting are the avenues of folding and unfolding that have not yet been explored or even conceived.

Acknowledgments

I thank my co-authors with whom I have enjoyed collaborating in the area of folding and unfolding.

References

[ABC+00] H.-K. Ahn, P. Bose, J. Czyzowicz, N. Hanusse, E. Kranakis, and P. Morin. Flipping your lid. *Geombinatorics*, 10(2):57–63, 2000.

[ABD+00] E. M. Arkin, M. A. Bender, E. D. Demaine, M. L. Demaine, J. S. B. Mitchell, S. Sethia, and S. S. Skiena. When can you fold a map? Computing Research Repository cs.CG/0011026, Nov. 2000. `http://www.arXiv.org/abs/cs.CG/0011026`.

[ACD+00] O. Aichholzer, C. Cortés, E. D. Demaine, V. Dujmović, J. Erickson, H. Meijer, M. Overmars, B. Palop, S. Ramaswami, and G. T. Toussaint. Flipturning polygons. In *Proc. Japan Conf. Discrete Comput. Geom.*, Lecture Notes in Comput. Sci., Tokyo, Japan, Nov. 2000. To appear in *Discrete and Computational Geometry*.

[ADE+01] O. Aichholzer, E. D. Demaine, J. Erickson, F. Hurtado, M. Overmars, M. A. Soss, and G. T. Toussaint. Reconfiguring convex polygons. *Comput. Geom. Theory Appl.*, 2001. To appear.

[AGP99] B. Aronov, J. E. Goodman, and R. Pollack. Convexification of planar polygons in $\mathbb{R}^3$. Manuscript, Oct. 1999. `http://www.math.nyu.edu/faculty/pollack/convexifyingapolygon10-27-99.ps`.

[AO92] B. Aronov and J. O'Rourke. Nonoverlap of the star unfolding. *Discrete Comput. Geom.*, 8(3):219–250, 1992.

[BDD+98] T. Biedl, E. Demaine, M. Demaine, A. Lubiw, M. Overmars, J. O'Rourke, S. Robbins, and S. Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canadian Conf. Comput. Geom.*, Montréal, Canada, Aug. 1998. `http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-biedl-unfolding.ps.gz`.

[BDD+99] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, M. Overmars, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides. Locked and unlocked polygonal chains in 3D. Technical Report 060, Smith College, 1999. A preliminary version appeared in the *Proc. 10th ACM-SIAM Sympos. Discrete Algorithms*, Baltimore, Maryland, Jan. 1999, pages 866–867.

[BDD+01] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides. A note on reconfiguring tree linkages: Trees can lock. *Discrete Appl. Math.*, 2001. To appear.

[BDE+01] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, and J. Snoeyink. Ununfoldable polyhedra with convex faces. *Comput. Geom. Theory Appl.*, 2001. To appear.

[BDEH98] M. Bern, E. Demaine, D. Eppstein, and B. Hayes. A disk-packing algorithm for an origami magic trick. In *Proc. Internat. Conf. Fun with Algorithms*, Isola d'Elba, Italy, June 1998.

[BDL+99] T. C. Biedl, E. D. Demaine, S. Lazard, S. M. Robbins, and M. A. Soss. Convexifying monotone polygons. In *Proc. Internat. Symp. Algorithms and Computation*, volume 1741 of *Lecture Notes in Comput. Sci.*, pages 415–424, Chennai, India, Dec. 1999.

[BH96] M. Bern and B. Hayes. The complexity of flat origami. In *Proc. 7th ACM-SIAM Sympos. Discrete Algorithms*, pages 175–183, Atlanta, Jan. 1996.

[CDR00] R. Connelly, E. D. Demaine, and G. Rote. Straightening polygonal arcs and convexifying polygonal cycles. In *Proc. 41st IEEE Sympos. Found. Comp. Sci.*, pages 432–442, Redondo Beach, California, Nov. 2000.

[CJ98] J. Cantarella and H. Johnston. Nontrivial embeddings of polygonal intervals and unknots in 3-space. *J. Knot Theory Ramifications*, 7(8):1027–1039, 1998.

[CKM+01] J. A. Calvo, D. Krizanc, P. Morin, M. Soss, and G. Toussaint. Convexifying polygons with simple projections. *Infor. Process. Lett.*, 2001. To appear.

[CO99] R. Cocan and J. O'Rourke. Polygonal chains cannot lock in 4D. In *Proc. 11th Canadian Conf. Comput. Geom.*, Vancouver, Canada, Aug. 1999. `http://www.cs.ubc.ca/conferences/CCCG/elec_proc/c17.ps.gz`.

[DD01] E. D. Demaine and M. L. Demaine. Recent results in computational origami. In *Proc. 3rd Internat. Meeting of Origami Science, Math, and Education*, Monterey, California, March 2001. To appear.

[DDL98] E. D. Demaine, M. L. Demaine, and A. Lubiw. Folding and cutting paper. In J. Akiyama, M. Kano, and M. Urabe, editors, *Revised Papers from the Japan Conf. Discrete Comput. Geom.*, volume 1763 of *Lecture Notes in Comput. Sci.*, pages 104–117, Tokyo, Japan, Dec. 1998.

[DDL00] E. D. Demaine, M. L. Demaine, and A. Lubiw. Flattening polyhedra. Manuscript, 2000.

[DDLO00a] E. Demaine, M. Demaine, A. Lubiw, and J. O'Rourke. Examples, counterexamples, and enumeration results for foldings and unfoldings between polygons and polytopes. Technical Report 069, Smith College, Northampton, MA, July 2000.

[DDLO00b] E. D. Demaine, M. L. Demaine, A. Lubiw, and J. O'Rourke. Enumerating foldings and unfoldings between polygons and polytopes. In *Proc. Japan Conf. Discrete Comput. Geom.*, Lecture Notes in Comput. Sci., Tokyo, Japan, Nov. 2000.

[DDM00] E. D. Demaine, M. L. Demaine, and J. S. B. Mitchell. Folding flat silhouettes and wrapping polyhedral packages: New results in computational origami. *Comput. Geom. Theory Appl.*, 16(1):3–21, 2000.

[Dür77] A. Dürer. *The Painter's Manual: A Manual of Measurement of Lines, Areas, and Solids by Means of Compass and Ruler Assembled by Albrecht Dürer for the Use of All Lovers of Art with Appropriate Illustrations Arranged to be Printed in the Year MDXXV.* Abaris Books, Inc., New York, 1977. English translation of *Unterweysung der Messung mit dem Zirkel un Richtscheyt in Linien Ebnen uhnd Gantzen Corporen*, 1525.

[ELR+98] H. Everett, S. Lazard, S. Robbins, H. Schröder, and S. Whitesides. Convexifying star-shaped polygons. In *Proc. 10th Canadian Conf. Comput. Geom.*, Montréal, Canada, Aug. 1998. `http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-everett-convexifying.ps.gz`.

[Erd35] P. Erdős. Problem 3763. *Amer. Math. Monthly*, 42:627, 1935.

[FHM+01] T. Fevens, A. Hernandez, A. Mesa, M. Soss, and G. Toussaint. Simple polygons that cannot be deflated. *Beiträge Algebra Geom.*, 2001. To appear.

[Gar83] M. Gardner. The combinatorics of paper folding. In *Wheels, Life and Other Mathematical Amusements*, chapter 7, pages 60–73. W. H. Freeman and Company, 1983.

[Grü95] B. Grünbaum. How to convexify a polygon. *Geombinatorics*, 5:24–30, July 1995.

[Hul94] T. Hull. On the mathematics of flat origamis. *Congr. Numer.*, 100:215–224, 1994.

[Jus94] J. Justin. Towards a mathematical theory of origami. In K. Miura, editor, *Proc. 2nd Internat. Meeting of Origami Science and Scientific Origami*, pages 15–29, Otsu, Japan, November–December 1994.

[Kaw89] T. Kawasaki. On the relation between mountain-creases and valley-creases of a flat origami. In H. Huzita, editor, *Proc. 1st Internat. Meeting of Origami Science and Technology*, pages 229–237, Ferrara, Italy, Dec. 1989. An unabridged Japanese version appeared in *Sasebo College of Technology Report*, 27:153–157, 1990.

[KM95] M. Kapovich and J. Millson. On the moduli space of polygons in the Euclidean plane. *J. Differential Geom.*, 42(1):133–164, 1995.

[Lan96] R. J. Lang. A computational algorithm for origami design. In *Proc. 12th Sympos. Comput. Geom.*, pages 98–105, Philadelphia, PA, May 1996.

[LO96] A. Lubiw and J. O'Rourke. When can a polygon fold to a polytope? Technical Report 048, Smith College, June 1996.

[Lun71] W. F. Lunnon. Multi-dimensional map-folding. *The Computer Journal*, 14(1):75–80, Feb. 1971.

[LW95] W. J. Lenhart and S. H. Whitesides. Reconfiguring closed polygonal chains in Euclidean d-space. *Discrete Comput. Geom.*, 13:123–140, 1995.

[MMP87] J. S. B. Mitchell, D. M. Mount, and C. H. Papadimitriou. The discrete geodesic problem. *SIAM J. Comput.*, 16(4):647–668, Aug. 1987.

[Mon91] J. Montroll. *African Animals in Origami.* Dover Publications, 1991.

[Nag39] B. Nagy. Solution to problem 3763. *Amer. Math. Monthly*, 46:176–177, Mar. 1939.

[O'R98] J. O'Rourke. Folding and unfolding in computational geometry. In *Revised Papers from the Japan Conf. Discrete Comput. Geom.*, volume 1763 of *Lecture Notes in Comput. Sci.*, pages 258–266, Tokyo, Japan, Dec. 1998.

[Sal73] G. T. Sallee. Stretching chords of space curves. *Geom. Dedicata*, 2:311–315, 1973.

[Sch89] C. Schevon. *Algorithms for Geodesics on Polytopes.* PhD thesis, Johns Hopkins University, 1989.

[She75] G. C. Shephard. Convex polytopes with convex nets. *Math. Proc. Cambridge Philos. Soc.*, 78:389–403, 1975.

[Str00] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st IEEE Sympos. Found. Comp. Sci.*, pages 443–453, Redondo Beach, California, Nov. 2000.

[Tou99a] G. Toussaint. Computational polygonal entanglement theory. In *Proceedings of the VIII Encuentros de Geometria Computacional*, Castellon, Spain, July 1999.

[Tou99b] G. Toussaint. The Erdős-Nagy theorem and its ramifications. In *Proc. 11th Canadian Conf. Comput. Geom.*, Vancouver, Canada, Aug. 1999. `http://www.cs.ubc.ca/conferences/CCCG/elec_proc/fp19.ps.gz`.

[Tou01] G. Toussaint. A new class of stuck unknots in pol_6. *Beiträge Algebra Geom.*, 2001. To appear.

[Whi92] S. Whitesides. Algorithmic issues in the geometry of planar linkage movement. *Australian Computer Journal*, 24(2):42–50, May 1992.