Even $1 \times n$ Edge-Matching and Jigsaw Puzzles are Really Hard

(Extended Abstract)

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Jigsaw puzzles [9] and edge-matching puzzles [5] are two ancient types of puzzle, going back to the 1760s and 1890s, respectively. Jigsaw puzzles involve fitting together a given set of pieces (usually via translation and rotation) into a desired shape (usually a rectangle), often revealing a known image or pattern. The pieces are typically squares with a pocket cut out of or a tab attached to each side, except for boundary pieces which have one flat side and corner pieces which have two flat sides. Most jigsaw puzzles have unique tab/pocket pairs that fit together, but we consider the generalization to "ambiguous mates" where multiple tabs and pockets have the same shape and are thus compatible.

Edge-matching puzzles are similar to jigsaw puzzles: they too feature square tiles, but instead of pockets or tabs, each edge has a color or pattern. In *signed* edge-matching puzzles, the edge labels come in complementary pairs (e.g., the head and tail halves of a colored lizard), and adjacent tiles must have complementary edge labels on their shared edge (e.g., forming an entire lizard of one color). This puzzle type is essentially identical to jigsaw puzzles, where complementary pairs of edge labels act as identically shaped tab/pocket pairs. In *unsigned* edge-matching puzzles, edge labels are arbitrary, and the requirement is that adjacent tiles must have identical edge labels. In both cases, the goal is to place (via translation and rotation) the tiles into a target shape, typically a rectangle.

A recent popular (unsigned) edge-matching puzzle is *Eternity II* [8], which featured a US$2,000,000 prize for the first solver (before 2011). The puzzle remains unsolved (except presumably by its creator, Christopher Monckton). The best partial solution to date [7] either places 247 out of the 256 pieces without error, or places all 256 pieces while correctly matching 467 out of 480 edges.

Previous work. The first study of jigsaw and edge-matching puzzles from a computational complexity perspective proved NP-hardness [2]. Four years later, unsigned edge-matching puzzles were proved NP-hard even for a target shape of a $1 \times n$ rectangle [3]. There is a simple reduction from unsigned edge-matching puzzles to signed edge-matching/jigsaw puzzles [2], which expands the puzzle by a factor of two in each dimension, thereby establishing NP-hardness of $2 \times n$ jigsaw puzzles. Unsigned $2 \times n$ edge-matching puzzles were claimed to be APX-hard [1], but the proof is incorrect.[1]

Our results. We prove that $1 \times n$ jigsaw puzzles and $1 \times n$ edge-matching puzzles are both NP-hard, even to approximate within a factor of 0.9999999762 ($> \frac{41899199}{41899200}$). This is the first correct inapproximability result for either problem. Even NP-hardness is new for $1 \times n$ signed edge-matching/jigsaw puzzles. By a known reduction [2], these results imply NP-hardness for polyomino packing (exact packing of a given set of polyominoes into a given rectangle) when the polyominoes all have area $\Theta(\log n)$; the previous NP-hardness proof [2] needed polyominoes of area $\Theta(\log^2 n)$.

We prove inapproximability for two different optimization versions of the problems. First, we consider placing the maximum number of tiles without any violations of the matching constraints. This objective has a simple $\frac{1}{2}$-approximation for $1 \times n$ puzzles: alternate between placing a tile and leaving a blank. Second, we consider placing all of the tiles while maximizing the number of compatible edges between

[1]Personal communication with Antonios Antoniadis, October 2014. In particular, Lemma 3's proof is incomplete.

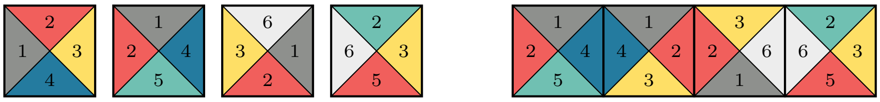

Figure 1: An example of a 1×4 (unsigned) edge-matching puzzle consisting of 4 tiles (left), where all tiles can be assembled into a 1×4 rectangular grid, with matching colors on the edges of adjacent tiles (right).

adjacent tiles (as in [1]).[2] This objective also has a simple $\frac{1}{2}$-approximation for $1 \times n$ puzzles, via a maximum-cardinality matching on the tiles in a graph where edges represent having any compatible edges: any solution with k compatibilities induces a matching of size at least $k/2$ [1]. Thus, up to constant factors, we resolve the approximability of these puzzles.

Our NP-hardness reduction is from Hamiltonian path on directed graphs whose vertices each has maximum in-degree and out-degree 2, which was shown to be NP-complete by Plesník [6]. To prove inapproximability, we reduce from a maximization version of this Hamiltonicity problem, called *maximum vertex-disjoint path cover*, where the goal is to choose as many edges as possible to form vertex-disjoint paths. (A Hamiltonian path would be an ideal path cover, forming a single path of length $|V| - 1$.) This problem is known to be NP-hard to approximate within some constant factor [4].

We prove that maximum vertex-disjoint path cover satisfies a stronger type of hardness, called *gap hardness*: it is NP-hard to distinguish between a directed graph having a Hamiltonian path versus one where all vertex-disjoint path covers having at most $0.999999284\,|V|$ ($> \frac{1396639}{1396640}\,|V|$) edges, given a promise that the graph falls into one of these two categories. This gap hardness immediately implies inapproximability within a factor of 0.999999284, although this constant is weaker than the known inapproximability [4]. More useful is that our reduction to $1 \times n$ jigsaw/edge-matching puzzles is gap-preserving, implying gap hardness and inapproximability for the latter. This approach lets us focus on "perfect" instances (where all tiles are compatible) versus "very bad" instances (where many tiles are incompatible), which seems far easier than standard L-reductions used in most inapproximability results, where we must distinguish between an arbitrary optimal and a factor below that arbitrary optimal.

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[2] A dual objective would be to place all tiles while minimizing the number of mismatched edges, but this problem is already NP-hard to distinguish between an answer of zero and positive, so it cannot be approximated.