Additive Approximation Algorithms for List-Coloring Minor-Closed Class of Graphs

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Abstract

It is known that computing the list chromatic number is harder than computing the chromatic number (assuming NP $\neq$ coNP). In fact, the problem of deciding whether a given graph is f-list-colorable for a function $f : V \to \{c-1, c\}$ for $c \geq 3$ is Π_2^p-complete. In general, it is believed that approximating list coloring is hard for dense graphs.

In this paper, we are interested in sparse graphs. More specifically, we deal with nontrivial minor-closed classes of graphs, i.e., graphs excluding some K_k minor. We refine the seminal structure theorem of Robertson and Seymour, and then give an additive approximation for list-coloring within $k-2$ of the list chromatic number. This improves the previous multiplicative $O(k)$-approximation algorithm [20]. Clearly our result also yields an additive approximation algorithm for graph coloring in a minor-closed graph class. This result may give better graph colorings than the previous multiplicative 2-approximation algorithm for graph coloring in a minor-closed graph class [6].

Our structure theorem is of independent interest in the sense that it gives rise to a new insight on well-connected H-minor-free graphs. In particular, this class of graphs can be easily decomposed into two parts so that one part has bounded treewidth and the other part is a disjoint union of bounded-genus graphs. Moreover, we can control the number of edges between the two parts. The proof method itself tells us how knowledge of a local structure can be used to gain a global structure, which gives new insight on how to decompose a graph with the help of local-structure information.

1 Introduction

1.1 Coloring, List Coloring, and Our Main Results.

Graph coloring is arguably the most popular subject in graph theory. Also, it is one of the central problems in combinatorial optimization, because it is one of the hardest problems to approximate. In general, the chromatic number is inapproximable in polynomial time within factor $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless coRP $=$ NP; see Feige and Kilian [12] and Håstad [15]. Even for 3-colorable graphs, the best known polynomial approximation algorithm [4] achieves a factor of $O(n^{0.211})$.

An interesting variant of the classic problem of properly coloring the vertices of a graph with the minimum possible number of colors arises when one imposes some restrictions on the colors or the number of colors available to particular vertices. This subject of *list coloring* was first introduced in the second half of the 1970s, by Vizing [37] and independently by Erdős, Rubin, and Taylor [11]. List coloring has since received considerable attention by many researchers, and has led to several beautiful conjectures and results.

Let us formally define list coloring. If $G = (V, E)$ is a graph, and f is a function that assigns a positive integer $f(v)$ to each vertex of v in G, we say that G is *f-choosable* (or *f-list-colorable*) if, for every assignment of sets of integers $S(v) \subseteq \mathbb{Z}$, where $|S(v)| = f(v)$ for all $v \in V(G)$, there is a proper vertex coloring $c : V \to \mathbb{Z}$ so that $c(v) \in S(v)$ for all $v \in V(G)$. Let L be a set of colors, and let $L(v)$ be a subset of L for each vertex v of G. An *L-coloring* of the graph G is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, and for every edge uv we have $c(u) \neq c(v)$. Such a coloring is called a *list coloring*. So list coloring is close to graph coloring, but each vertex has its own list, and coloring must use one color in the list of each vertex. The smallest integer k such that G is f-choosable for $f(v) = k$ ($v \in V(G)$) is the *list chromatic number* $\chi_l(G)$. If G is f-choosable for $f(v) = s$ ($v \in V(G)$), we sometimes say that G has a list coloring using at most s colors.

Clearly, $\chi(G) \leq \chi_l(G)$, and there are many graphs for which $\chi(G) < \chi_l(G)$. A simple example is the complete bipartite graph $K_{2,4}$, which is not 2-choosable. Another well-known example is the complete bipartite graph $K_{3,3}$. In fact, it is easy to show that for every k, there exists a bipartite graph whose list chromatic number is bigger than k.

The problem of computing the list chromatic number of a given graph is therefore difficult, even for small graphs with a simple structure. It is shown in [13] that the problem of deciding whether a given graph is f-list-colorable for a function $f : V \to \{k-1, k\}$ for $k \geq 3$ is Π_2^p-complete. Hence, under the common assumption NP $\neq$ coNP, the problem is strictly harder than the NP-complete problem of deciding whether the chromatic number is k (if $k \geq 3$).

Let us highlight some difficulties between list coloring and graph coloring. There is an approximation technique for graph coloring using semidefinite programming [16], but this does not seem to extend to list coloring. Also, a simple combinatorial algorithm for graph coloring to detect a large independent set to give the same color does not apparently

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work.

Another difficulty comes in terms of "density" of graphs. Graph coloring is trivial for bipartite graphs, but it is actually NP-hard to determine the list chromatic number of bipartite graphs; see [34]. In general, it is believed that approximate list coloring is hard for dense graphs. For more details, we refer the reader to Alon [1]. Therefore, it would be very interesting to consider sparse graphs.

Although there are many negative results as stated above, there are some positive results, which are mainly connected to the Four Color Theorem. One celebrated example is Thomassen's result on planar graphs [32]. It says that every planar graph is 5-choosable, and its proof is within 20 lines and gives rise to a linear-time algorithm to 5-list-color planar graphs. In contrast with the Four Color Theorem, there are planar graphs that are not 4-choosable [36]. These were conjectured by Erdős, Rubin and Taylor [11].

It is well-known that planar graphs are closed under taking minor operations; that is, deleting edges, deleting vertices and contracting edges. So one natural question is whether we can extend the result of Thomassen to more general minor-closed graph classes. Before we mention our main result, let us first discuss list coloring of planar graphs and bounded-genus graphs.

Thomassen's result [32] can be rephrased algorithmically as follows:

THEOREM 1.1. [32] *There is a linear-time algorithm to list-color a planar graph G using at most $\chi_l(G) + 2$ colors.*

To clarify the meaning of Theorem 1.1, the algorithm behaves as follows:

1. The algorithm first outputs a number $c \leq \chi_l(G) + 2$.
2. Then, for any given lists L with each vertex having at least c colors, the algorithm gives an L-coloring.

In this paper, whenever we speak of an additive approximation algorithm within t of the list chromatic number, we mean the above two points (with "$+2$" will be replaced by $+t$).

The bound $\chi_l(G) + 2$ is essentially best possible for planar graphs, because it is NP-hard to decide whether or not they are 4-list-colorable, and they are 3-list-colorable [34]. In fact, the problem of deciding whether a given planar graph is f-list-colorable for the constant function $f : V \to \{3\}$ or $f : V \to \{4\}$ is Π_2^p-complete, as proved by Gutner [13]. This implies that distinguishing between 3-, 4-, and 5-list-colorability is Π_2^p-complete for planar graphs and bounded-genus graphs. This also means that, unless $\mathrm{coRP} = \mathrm{NP}$, one cannot approximate the list chromatic number of planar graphs and bounded-genus graphs within 1 of the list chromatic number $\chi_l(G)$.

In [18], Kawarabayashi develops an additive approximation algorithm for list-coloring bounded-genus graphs within 2 of the list chromatic number, which is best possible.

THEOREM 1.2. [18] *Suppose G is embedded on a fixed surface. Then there is a linear-time algorithm to list-color the graph using at most $\chi_l(G) + 2$ colors.*

So, in terms of approximation algorithms, the case of bounded-genus graphs is solved. One natural question is to extend Theorem 1.2 to a general minor-closed class of graphs, i.e., graphs excluding some K_k minors. Coloring and list-coloring a minor-closed class of graphs are interesting in a mathematical sense because they are connected to the well-known Hadwiger's conjecture [14]. The problem is interesting in theoretical computer scientists because we can get good approximation algorithms, e.g., [6]. Let us emphasize that approximating the chromatic number of a general graph is known to be a very hard problem. As we discussed, list coloring is known to be even harder than graph coloring. So it would be an interesting question to get a good approximation algorithm for a minor-closed class of graphs comparable to the graph-coloring case obtained in [6].

Our main theorem is the following.

THEOREM 1.3. *Let $\mathcal{M}$ be a minor-closed class of graphs and suppose that some graph of order k is not a member of $\mathcal{M}$. Then there is a polynomial-time algorithm for list-coloring graphs in $\mathcal{M}$ with $\chi_l(G) + k - 2$ colors.*

This improves the multiplicative $O(k)$-approximation algorithm by Kawarabayashi and Mohar [20]. Improving the additive approximation in Theorem 1.3 would be difficult. In Section 7, we highlight some technical difficulties. We would probably need some structure theorem significantly generalizing the seminal Robertson-Seymour decomposition theorem.

To prove Theorem 1.3, we refine the seminal structure theorem of Robertson and Seymour [28] to well-connected graphs. In particular, this gives rise to a new insight of the decomposition theorem in the sense that this class of graphs can be easily decomposed into two parts so that one part has bounded treewidth and the other part is a disjoint union of bounded-genus graphs. Moreover, we can control the number of edges between two parts.

The proof method itself tells us how knowledge of a local structure can be used to gain a global structure, which gives new insight on how to decompose a graph with the help of local-structure information.

1.2 Remarks. Let us address the difference between graph coloring and list coloring of minor-closed class of graphs. There is a 2-approximation algorithm for graph coloring of minor-closed class of graphs [6], but it seems that there is a huge gap between list coloring and graph coloring of minor-closed class of graphs. The 2-approximation algorithm in [6] follows from the following result.

THEOREM 1.4. [6] *In polynomial time, every H-minor-free graph can be partitioned into two vertex sets V_1, V_2 such that both V_1 and V_2 have treewidth at most $f(H)$ for some function depending on $|H|$.*

This theorem may not be useful for list coloring. This is because some vertex v in V_1 may have a lot of neighbors in V_2, say at least $\chi_l(G)^2$ neighbors. So $\chi_l(G)^2$ colors in the list of v may be used in the list coloring of V_2.

Theorem 1.3 also gives rise to an additive approximation algorithm for graph-coloring a minor-closed class of graphs, i.e., K_k-minor-free graphs. In fact, we can improve the additive approximation from $k-2$ to $k-3$ in Theorem 1.3 for graph coloring. This will be discussed in Section 6. As far as we see, this additive approximation algorithm is not implied by the 2-approximation algorithm. Actually, if the chromatic number is at least $k-2$, Theorem 1.3 gives rise to a better approximation. Currently, the best known result for the chromatic number of graphs without K_k minors is $O(k\sqrt{\log k})$ by the theorem obtained independent by Thomason [31] and Kostochka [22]. Therefore, these two algorithms are comparable at the moment. Interestingly, the famous and well-known conjecture of Hadwiger [14] (which says that every graph without K_k minors is $(k-1)$-colorable) implies that the 2-approximation result would "almost" cover Theorem 1.3.[1]

1.3 Overview of the Algorithm.

Our main result is based on the seminal Robertson-Seymour Graph Minor decomposition theorem [28] together with a new technique that is also developed in the Graph Minor series. We also use the precoloring technique, which was developed by Thomassen [32] for planar graphs.

Roughly, the decomposition theorem represents any H-minor-free graph as a tree of clique-sums of graphs that are almost-embeddable into bounded-genus surfaces with bounded number of apex vertices; see Appendix A for definitions. We refer to the almost-embeddable graphs as "pieces" or "bags". A polynomial-time algorithm constructs such a tree decomposition [6]. Hereafter, we assume that the tree decomposition is given.

We now fix the root of the tree decomposition, and we allow precoloring at most $k-2$ vertices in the apex vertex set of the root. We extend this precoloring to an L-coloring of the whole graph using $\chi_l(G)+k-2$ colors. Our list-coloring algorithm proceeds as follows:

We proceed by induction on the number of vertices. If there are two bags B_s, B_t in the tree decomposition such that B_s is a parent of B_t and $|B_s \cap B_t| \leq k-2$, then we split the decomposition at $B_s \cap B_t$ into two tree decompositions T_1, T_2 such that B_t is the root of the tree decomposition T_2. The decomposition theorem guarantees that $B_s \cap B_t$ is contained in the apex vertex set of B_t.

By induction, we can list-color the vertices in T_1 by extending the precoloring in the root. Therefore, the list coloring of T_1 gives rise to a precoloring of $B_s \cap B_t$. This precoloring together with the tree decomposition T_2 satisfies the induction hypothesis. So we can list-color the vertices of T_2 by extending the precoloring of $B_s \cap B_t$. To clarify how the lists L are given, we first output numbers c_1, c_2 with $c_1 \leq \chi_l(B_s)+k-2$ and $c_2 \leq \chi_l(B_t)+k-2$, respectively. Then we output the maximum of c_1, c_2, say c_2. After that, we are given the lists L with each vertex having at least c_2 colors, and we L-color the whole graph G.

Hence, we may assume that there are no such separations in the decomposition. This assumption allows us to modify the decomposition theorem in the following way:

1. By adding a bounded number of vortices, each vertex in the surface part (of each bag) has at most $k-5$ neighbors in the apex vertex set, and it does not involve any clique-sum.

2. Each vortex is far apart from any other.

3. If we delete the vortices and the apex vertex set from each bag, then the resulting graph W is a disjoint union of bounded-genus graphs. Moreover, $G-W$ has bounded treewidth.

The third condition allows us to list-color the bounded treewidth graph $G-W$ using at most $p \leq \chi_l(G)$ colors by the standard dynamic programming. Thus at the moment, we can output the number $p+k-2 \leq \chi_l(G)+k-2$. We now assume that we are given lists L with each vertex having at least $p+k-2$ colors.

As discussed in the previous section, we may assume that $\chi_l(G) \geq 3$ (and hence $p \geq 3$). So the first condition implies that, for each bag, no matter how we color the apex vertices, each vertex in the surface part still has six available colors in its list (because we list-color the graph using at most $\chi_l(G)+k-2$ colors). This allows us to use the result by DeVos, Kawarabayashi, and Mohar [8]. Specifically, the following is possible.

Suppose a graph G consists of a graph embedded on a fixed surface with large representativity, together with a bounded number of vortices. Suppose furthermore that each of the vortices is far apart from any other. Suppose all the vortices are precolored using at most $\chi_l(G)$ colors, i.e., the precoloring of the vortices uses only the colors in the list of each vertex in the vortices whose order is at most $\chi_l(G)$. If each vertex in the surface has at least $\chi_l(G)+3 \geq 6$ available colors in its list, and sees at most $\chi_l(G)-1$ colors under the precoloring of the vortices, then the precoloring of the vortices can be extended to an L-coloring of the whole graph G using at most $\chi_l(G)+3$ colors. The same conclusion holds if we replace $\chi_l(G)$ by any number $p \geq 3$.

This result allows us to list-color the graph W (in 3) using at most $\chi_l(G)+3$ colors. Because each vertex in the surface sees at most $k-5$ vertices in the apex vertex set and $G-W$ has an L-coloring using at most $\chi_l(G)$ colors, we can list-color a given graph using at most $\chi_l(G)+3+k-5 = \chi_l(G)+k-2$ colors.

[1] Hadwiger's conjecture is trivially true for $k \leq 3$, and reasonably easy for $k=4$, as shown by Dirac [10] and Hadwiger himself [14]. However, for $k \geq 5$, Hadwiger's conjecture implies the Four Color Theorem. In 1937, Wagner [35] proved that the case $k=5$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [30] proved that a minimal counterexample to the case $k=6$ is a graph G which has a vertex v such that $G-v$ is planar. By the Four Color Theorem, this implies Hadwiger's conjecture for $k=6$. Hence the cases $k=5,6$ are each equivalent to the Four Color Theorem [2, 3, 24]. Hadwiger's conjecture is open for all $k \geq 7$. For the case $k=7$, Kawarabayashi and Toft [19] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. Recently, Kawarabayashi [17] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

The details of our modifications of the Robertson-Seymour decomposition theorem for each bag will be given in the full paper. The key idea is to understand the concept of "tangle", and its behavior in the surface of large representativity. The first one enables us to gain knowledge of a global structure from knowledge of its local structure relative to each tangle. The second one uses deep understandings of a graph embedded on a surface of large representativity [26], and a new metric of this family of graphs [27].

This paper is organized as follows. In Section 2, we give our main structure theorem, which modifies the seminal Robertson-Seymour decomposition theorem. Some formal definitions and some tools we need are given in Appendix A. In Section 3, we discuss list coloring of a bounded-genus graph. In Section 4, we show how to list-color the graph using the structure theorem. In Section 5, we give our main algorithm. In Section 6, we show how to improve the additive approximation in Theorem 1.3 from $k-2$ to $k-3$ for the usual graph-coloring case. Finally, in Section 7, we discuss some difficulties for improving the additive approximation in Theorem 1.3 for general H-minor-free graphs, and mention improvements for apex-minor-free graphs.

2 Refined Structure Theorem

We now state our structure theorem, which will be used in our proof. This theorem strengthens the seminal Robertson-Seymour decomposition theorem [28, Theorem 1.3] and a polynomial-time algorithm to find the decomposition [6] to add the three stated conditions. For the notations of h-almost-embeddable graphs, clique-sum, vortices, society vertices (of vortices), representativity, and Robertson-Seymour metric, see Appendix A.

THEOREM 2.1. *For any complete graph $H = K_k$, there is a constant h such that any H-minor-free graph can be written as clique $\leq h$-sums of h-almost-embeddable graphs such that the following three conditions hold:*

1. *In the surface part (which does not include the vertices contained in the vortices, i.e., society vertices) of each bag, there are no vertices that have at least $k-4$ neighbors in the apex vertex set.*

2. *If a bag B_i contains at most three (but at least one) vertices in the surface part of the parent bag B_{i-1}, then B_i contains at most $k-5$ vertices of the apex vertex set in B_{i-1}.*

3. *Let $D_1, D_2, \dots, D_h$ be vortices of a bag B. Then any two of $D_1, D_2, \dots, D_h$ have large distance (in a sense of Robertson-Seymour metric), actually distance at least q from any other, where q is a function of h. In other words, each of $D_1, D_2, \dots, D_h$ is far from any other. In addition, the representativity of the graph on a fixed surface in each bag is large, at least $r(h) \geq r(g,q,d)$ for some function of h, where $r(g,q,d)$ and g,q,d come from Theorem 3.1.*

Furthermore, there is a polynomial-time algorithm to construct this clique-sum decomposition for a given K_k-minor-free graph. The time complexity is $n^{O(h)}$.

3 List-Coloring Extensions in Bounded-Genus Graphs

We begin with some easy observations about list coloring.

Recall the definition of list coloring and its precoloring extension. If k is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then L is a *k-list-assignment*. The graph is *k-choosable* or *k-list-colorable* if it admits an L-coloring for every k-list-assignment L. If $L(v) = \{1, 2, \dots, k\}$ for every v, then every L-coloring is referred to as a *k-coloring* of G. If G admits an L-coloring (k-coloring), then we say that G is *L-colorable* (*k-colorable*). Suppose W is a subgraph of G, and each vertex of W has a list with at most $\chi_l(G)$ colors. If W is precolored, and this precoloring uses only the colors in the list of each vertex of W, we say that W *is precolored using at most $\chi_l(G)$ colors*.

In order to get our algorithm, we need to know what kind of graphs are 2-list-colorable. The following result gives the answer.

LEMMA 3.1. [11] *A graph is 2-choosable if and only if it is a bipartite graph plus additional structures which can be recognized in polynomial time.*

Let G be a graph embedded in the surface S. For $i = 1, 2, \dots, l$, let $C_i \subseteq V(G)$ be a set such that all vertices in C_i lie on the boundary of some face F_i, where $F_1, F_2, \dots, F_l$ are pairwise distinct faces. We call $C_1, C_2, \dots, C_l$ *cuffs*, because one can make them lie on distinct boundary components, after cutting holes in $F_1, F_2, \dots, F_l$.

One challenge for our algorithm is the following. Suppose the vertices in the bounded number of cuffs are precolored. Can we extend this precoloring to an L-coloring of the whole graph? To answer this question, we use the following tool developed by DeVos, Kawarabayashi and Mohar [8], which generalizes the graph-coloring case by Thomassen [33] to the list-coloring case. In fact, our statement below is different from the original, but it follows from the same proof as in [8] by combining with the Robertson-Seymour metric.

THEOREM 3.1. *For any three nonnegative integers g, q, d (with $q \geq 4$), there exists a natural number $r(g,q,d)$ such that the following holds. Suppose that G is embedded on a fixed surface S of Euler genus g and of the representativity at least $r(g,q,d)$, and there are d disjoint cuffs $S_1, S_2, \dots, S_d$ such that the distance (in the sense of the Robertson-Seymour metric) of any two cuffs of $S_1, S_2, \dots, S_d$ is at least q. Suppose furthermore that all the vertices in $S_1, S_2, \dots, S_d$ are precolored using at most $\chi_l(G)$ colors. Furthermore, the following conditions are satisfied:*

1. *all the faces except for the cuffs $S_1, S_2, \dots, S_d$ are triangles; and*

2. *each vertex in $G - (S_1 \cup S_2 \cup \dots \cup S_d)$ has at least $\chi_l(G) + 3$ colors in its list, and no vertex of $G - (S_1 \cup$*

$S_2 \cup \cdots \cup S_d$) is joined to more than $\chi_l(G) - 1$ colors under the precoloring of the cuffs $S_1, S_2, \ldots, S_d$.

Then the precoloring of the cuffs $S_1, S_2, \ldots, S_d$ using at most $\chi_l(G)$ colors can be extended to an L-coloring of G using at most $\chi_l(G) + 3$ colors. In fact, in the surface part, we need only six colors for all of vertices. (Precoloring may use more than six colors.) Also, there is a polynomial-time algorithm for such an L-coloring of G, given a precoloring for the cuffs $S_1, S_2, \ldots, S_d$, and the lists L for each vertex of G. The same conclusion holds if we replace $\chi_l(G)$ by any number $p \geq 3$.

As we said, the proof of Theorem 3.1 is almost identical to that in [8], but let us give some intuition. The assumption of Theorem 3.1 implies that after deleting all the vertices in $S_1 \cup S_2 \cup \ldots, \cup S_d$, there are exactly d cuffs $S'_1, S'_2, \ldots, S'_d$ such that each vertex in $S'_1 \cup S'_2 \cup \ldots, \cup S'_d$ has a list with at least four available colors that are not used in its colored neighbors of the precoloring of the cuffs $S_1, S_2, \ldots, S_d$. And every vertex not on these cuffs has a list with at least six available colors. The result in [8] says that the resulting graph is even 5-list-colorable. But if we allow every vertex not on the cuffs to have a list with at least six available colors, then the proof becomes much easier even if all the vertices on the cuffs have a list with only four available colors. So, we refer the reader to the proof of [8].

Theorem 3.1 is one of the keys in our algorithm.

Our proof gives rise to an additive approximation algorithm for graph coloring within $k - 3$ of the chromatic number. In this case, we need to improve Theorem 3.1; see Section 6. Actually, we only need the graph-coloring version of the improvement of Theorem 3.1, which was already proved in [33].

4 List-Coloring the Clique-Sum Decomposition of h-Almost Embeddable Graphs

Suppose we are given the Robertson-Seymour clique-sum decomposition of G. So G has pieces $B_1, B_2, \ldots$ such that each B_i has an h-almost-embeddable structure. Set B_1 to be the root of this clique-sum decomposition. For each piece B_i, define a graph G'_i the surface part of B_i. Also, let X_i be the apex vertex set of B_i. Our approximation algorithm first outputs a number $p \leq \chi_l(G)$, and then given any lists L with each vertex having at least $p + k - 2$ colors, outputs an L-coloring.

Our main idea is to use the structure theorem in Theorem 2.1, together with the precoloring technique developed by Thomassen [32]. More precisely, we are now allowed to precolor $k - 2$ vertices in the apex vertex set X_1 of B_1. Suppose that the vertices of X' in X_1 with $|X'| = k - 2$ are precolored using colors $1, 2, \ldots, k - 2$. (The precoloring may not use all of these colors, but in this case, the proof is easier, so we assume that $|X'| = k - 2$ and the vertices in X' are precolored using the colors $1, \ldots, k - 2$.) We will extend this precoloring to an L-coloring of the whole graph.

As we pointed out before, we may assume that the list chromatic number of G is at least 3; otherwise, we are able to list-color the whole graph using Lemma 3.1. Our proof is by induction on the number of bags. In the following arguments, we sometimes say that treewidth is "bounded". This means that the treewidth is at most $f(h)$ for some function of h.

Suppose for now that there is only one piece B_1. For simplicity, let G' be the surface part of B_1.

4.1 List-Coloring an h-Almost-Embeddable Graph. We partition the vertices of the surface part G' into two parts F_1, F_2 such that each vertex in F_1 has at least $k - 4$ neighbors, and all other vertices are in F_2.

By Theorem 2.1, we now have at most h disks $D'_1, D'_2, \ldots, D'_h$ such that all vertices in F_1 are in the graphs consisting of the union of the graphs embedded inside D'_i for $i = 1, 2, \ldots, h$ (actually, these graphs are vortices), and the distance (in the sense of the Robertson-Seymour metric) between any two of the disks $D'_1, D'_2, \ldots, D'_h$ is at least $q \geq 4$. Let us observe that by our assumption, Condition 2 of Theorem 2.1 does not happen. We will now add some of vertices in the surface to these disks $D'_1 \cup D'_2 \cup \cdots \cup D'_h$.

Let L be the vertices in the surface G' such that each vertex in L has at least 3 neighbors to one of disks $D'_1, D'_2, \ldots, D'_h$. Note that no vertex has neighbors in any two of the disks $D'_1, D'_2, \ldots, D'_h$ because there are no two disks of the distance (in the sense of the Robertson-Seymour metric) at most $q \geq 4$.

Let D''_i denote the graph in the disk D'_i. Let $N_i = (B_1 - G') \cup D''_1 \cup D''_2 \cup \cdots \cup D''_i \cup L$. Let $N = \bigcup_{i=1}^h N_i$. We claim that N has bounded treewidth. We prove this by induction on i. Clearly the apex vertex set X has bounded treewidth. Therefore, the statement is true when $i = 0$. Let us observe that, because the distance between any two of the disks $D'_1, D'_2, \ldots, D'_h$ is at least $q \geq 4$, N is a tree decomposition. By [7, Lemma 3], for any two graphs G' and G'', $\mathrm{tw}(G' \oplus G'') \leq \max\{\mathrm{tw}(G'), \mathrm{tw}(G'')\}$ (where $\oplus$ denotes clique-sum). Therefore, if each D''_i, together with the vertices in L that have at least three neighbors in D''_i, has bounded treewidth, it follows from the above mentioned result that N has bounded treewidth.

To see why adding vertices in L to each vortex D''_i does not increase the treewidth so much, first note that clearly each vortex has bounded treewidth. If we add the vertices of L to each of the disks of $D'_1, D'_2, \ldots, D'_h$, then the maximum size of grid-minor in N_i would increase by at most a factor of 2, because we only add the vertices that are the first neighbors of each disk of $D'_1, D'_2, \ldots, D'_h$. Note that any vertex in L is adjacent to only one of the disks $D'_1, D'_2, \ldots, D'_h$ because any disk of $D'_1, D'_2, \ldots, D'_h$ has distance at least $q \geq 4$ from any other. So, the treewidth may increase, but it has increased by some constant factor depending on h. This follows from the min-max relation of the treewidth and the size of the grid minor [25, 23, 9, 29, 5]. So each vortex has bounded treewidth, even after adding vertices in L to each vortex. Hence it follows that N has bounded treewidth.

Because N has bounded treewidth, we can list-color N using at most $p \leq \chi_l(G)$ ($p \geq 3$) colors in linear time by using the coloring method in bounded-treewidth graphs

[34]. In fact, because X' is precolored, this coloring may use $p+k-2 \leq \chi_l(G)+k-2$, but $N-X'$ can be colored using at most $p \leq \chi_l(G)$ colors. So, at the moment, we output the number $p+k-2 \leq \chi_l(G)+k-2$. We now assume that we are given lists L with each vertex having at least $p+k-2$ colors, and N is L-colored using at most $p+k-2$ colors (but $N-X'$ is L-colored using p colors). Hereafter, we assume $p=\chi_l(G)$, because this is the most difficult case, and other cases are easily obtained by exactly the same argument.

So, right now, the vertices on the outer boundary of the disks $D'_1, D'_2, \ldots, D'_h$, the vertex set L, and the apex vertices X are precolored using at most $\chi_l(G)+k-2$ colors, and any disk of $D'_1, D'_2, \ldots, D'_h$ has distance $q \geq 4$ from any other. Note that the representativity of G' is at least $r(g,q,d)$, where $r(g,q,d)$ come from Theorem 3.1. Each vertex in G' that is outside the disks $D'_1, D'_2, \ldots, D'_h$ has at most $k-5$ neighbors in the apex vertex set X. Therefore, each vertex in $G'-L$ that is outside the disks $D'_1, D'_2, \ldots, D'_l$ has at least $\chi_l(G)+k-2-(k-5) \geq \chi_l(G)+3 \geq 6$ available colors in its list, which are not used in the coloring of its colored neighbors (because we are trying to list-color G' using at most χ_l+k-2 colors). Concerning the vertices in L, they receive one color from the coloring of N. In addition, they see at most $k-5$ vertices in the apex vertex set. Therefore, each of them has at least $(\chi_l(G)+k-2)-(\chi_l(G)-1+k-5)=4$ available colors in its list, which are not used in the coloring of its colored neighbors. In addition, it sees at most $\chi_l(G)-1$ colors under the coloring of the vortices. Therefore, by Theorem 3.1, we can list-color all the vertices in G' using $\chi_l(G)+3$ colors, and thus in G using at most $\chi_l(G)+3+k-5=\chi_l(G)+k-2$ colors. This completes the proof when there is exactly one bag. Let us observe that the argument here gives rise to list-coloring an h-almost-embeddable graph using at most $\chi_l(G)+k-2$ in polynomial time.

4.2 List-Coloring Clique-Sum Decomposition. We now assume that there is a clique-sum decomposition such that B_1 is the root, and $B_i \ldots$ are bags.

At the moment, we do not require the clique-sum decomposition to satisfy the three additional properties in Theorem 2.1.

If there are two pieces B_j and B_{j+1} such that B_j is a parent of B_{j+1}, and $|B_j \cap B_{j+1}| \leq k-2$, then this means that the clique-sum has a separation (A,B) such that $A \cap B = B_j \cap B_{j+1}$, $|A \cap B| \leq k-2$, and A contains B_j and B contains B_{j+1}. In addition, all the vertices in $B_j \cap B_{j+1}$ are contained in the apex vertex set of B_{j+1} (because otherwise we just need to add them to the apex vertex set). We can think of B_{j+1} as the root of the clique-sum decomposition of B. In this case, we first apply induction to A, and after list-coloring A, we apply induction to B with the precoloring of $A \cap B$ which comes from the coloring of A. Because $|A \cap B| \leq k-2$, and all the vertices of $A \cap B = B_j \cap B_{j+1}$ are contained in the apex vertex set of B_{j+1} (and B_{j+1} is the root of the clique-sum decomposition of B), the induction hypothesis is satisfied for B. To clarify how the lists L are given, we first output numbers c_1, c_2 with $c_1 \leq \chi_l(A)+k-2$ and $c_2 \leq \chi_l(B)+k-2$, respectively. Then we output the maximum of c_1, c_2, say c_2. After that, we are given the lists L with each vertex having at least c_2 colors, and we L-color the whole graph G.

Therefore, we may assume that there are no such separations in the clique-sum. We now apply Theorem 2.1. For each bag B_i, define a graph G'_i the surface part of B_i. We can define L_i as L in the previous subsection. Let $D'_{i,1}, D'_{i,2}, \ldots, D'_{i,h_i}$ denote the disks containing the vortices. By our separations property, Condition 2 of Theorem 2.1 does not happen. In particular, this implies that no vertex in the surface part G_i involves the clique-sums. We now partition the vertices of the surface part G'_i into two parts F_1, F_2 such that each vertex in F_1 has at least $k-4$ neighbors in the apex vertex set, and all other vertices are in F_2. By Theorem 2.1, all the vertices in F_1 are in the graph consisting of the union of the graphs embedded inside the disk $D'_{i,j}$ for $j=1,2,\ldots,h_i$ (actually, these graphs are vortices), and the distance (in the sense of the Robertson-Seymour metric) between any two of the disks $D'_{i,1}, D'_{j,2}, \ldots, D'_{i,h_i}$ is at least $q \geq 4$.

For each piece B_i, let N_i be the graph $(B_i - G'_i) \cup D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,h_i} \cup L_i$, where $D_{i,j}$ is the graph in the disk $D'_{i,j}$. Let $N = \bigcup_{i=1} N_i$.

Then we claim that N has bounded treewidth. We prove this by induction on i. Let us observe that N is a tree decomposition because no vertex of $N_i - L_i$ is involved in the surface part of any bag and each vertex of L_i is adjacent to exactly one disk of the disks $D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,h_i}$.

When $i=1$, the result follows from the previous section. We have proved that N_1 has bounded treewidth.

Suppose by induction that $N_1 \oplus N_2 \oplus \cdots \oplus N_i$ has bounded treewidth. Because N_{i+1} has bounded treewidth, as proved in the previous section, by the above remark, $N_1 \cup N_2 \cup \cdots \cup N_i \cup N_{i+1}$ also has bounded treewidth. Hence N has bounded treewidth.

Because N has bounded treewidth, we can list-color N using at most $p \leq \chi_l(G)$ $(p \geq 3)$ colors in linear time by using the coloring method in bounded-treewidth graphs [34]. In fact, because X' is precolored, this coloring may use $p+k-2 \leq \chi_l(G)+k-2$, but $N-X'$ can be colored using at most $p \leq \chi_l(G)$ colors. So, at the moment, we output the number $p+k-2 \leq \chi_l(G)+k-2$. We now assume that we are given lists L with each vertex having at least $p+k-2$ colors, and N is L-colored using at most $p+k-2$ colors (but $N-X'$ is L-colored using p colors). Hereafter, we assume $p=\chi_l(G)$, because this is the most difficult case, and other cases are easily obtained by the exactly same argument.

Delete N from G. Because no vertex in the surface part is involved in the clique-sum, so, the resulting graph consists of disjoint unions of graphs, each of which has a 2-cell embedding into a surface of bounded genus with representativity at least $r(g,q,d)$. Therefore, we can focus on one of the bounded-genus graphs G'_i, and then apply the same argument to all other bounded-genus graphs in the same way. For simplicity, we assume that G' is the surface part of some bag, and we focus on the graph G' (so hereafter, we omit the index i).

So, right now, vertices on the outer face boundary of the disks $D'_1, D'_2, \dots, D'_h$, the vertex set L, and the vertices in the apex vertex set X are precolored using at most $\chi_l(G) + k - 2$ colors, and any of the disks $D'_1, D'_2, \dots, D'_h$ has distance $q \geq 4$ from any other. Note that the representativity of G' is at least $r(g, q, d)$, where $r(g, q, d)$ comes from Theorem 3.1. Each vertex in G' that is outside the disks $D'_1, D'_2, \dots, D'_l$ has at most $k - 5$ neighbors in the apex vertex X.

Therefore, each vertex in $G' - L$ that is outside the disks $D'_1, D'_2, \dots, D'_l$ has at least $\chi_l(G) + k - 2 - (k - 5) \geq \chi_l(G) + 3 \geq 6$ available colors in its list, which are not used in the coloring of its colored neighbors (because we are trying to list-color G' using at most $\chi_l + k - 2$ colors). Concerning the vertices in L, they receive one color from the coloring of N. In addition, they see at most $k - 5$ vertices in the apex set. Therefore, each of them has at least $(\chi_l(G) + k - 2) - (\chi_l(G) - 1 + k - 5) = 4$ available colors in its list, which are not used in the coloring of its colored neighbors. In addition, it sees at most $\chi_l(G) - 1$ colors under the coloring of the vortices. Hence, by Theorem 3.1, we can list-color all the vertices in G' using at most $\chi_l(G) + 3$ colors, and thus G using at most $\chi_l(G) + 3 + k - 5 = \chi_l(G) + k - 2$ colors.

We can then apply the above argument to list-color all the bounded-genus graphs in $G - N$ using at most $\chi_l(G) + 3$ colors. This allows us to obtain a $(\chi_l(G) + k - 2)$-list-coloring of G. This completes the proof. Let us observe that the argument here gives rise to list-color a clique-sum decomposition of h-almost embeddable graphs using at most $\chi_l(G) + k - 2$ in polynomial time.

5 Algorithm

We are now ready to describe our algorithm. Let us assume that $q, r(g, q, d)$ are as in Theorem 3.1. In addition, h is as in Theorem 2.1. As mentioned above, we may assume $\chi_l(G) \geq 3$; otherwise, the problem is easy by Lemma 3.1. The Robertson-Seymour clique-sum decomposition can be constructed in polynomial time [6]. The time complexity is $n^{O(h)}$. So we assume that it is given.

Algorithm for Theorem 1.3

Input: A Robertson-Seymour clique-sum decomposition. In addition, at most $k - 2$ vertices (let us call them X') in the apex vertex set of the root of the clique-sum decomposition are precolored. Furthermore, the current graph G has $\chi_l(G) \geq 3$.

Output: As described in Theorem 1.3. Moreover, given any lists L with each vertex having at least $\chi_l(G) + k - 2$ colors, the algorithm gives an L-coloring of G.

Running time: $n^{O(h)}$.

Description:

Step 1. Finding a small separation of two bags.

We now test whether the clique-sum decomposition has a vertex set W with $|W| \leq k - 2$ such that $W = B_t \cap B_s$ for two bags B_s, B_t, where B_s is a parent of B_t. If such a separation exists, we just cut off W. Let us look at this argument more closely.

Let us start at the root bag B_1. Suppose $B_2, B_3, \dots, B_s$ are children of B_1, and $|B_i \cap B_1| \leq k - 2$ for $2 \leq i \leq s$. Then we delete all the bags $B_2 - B_1, B_3 - B_1, \dots, B_s - B_1$ in B_1. For $B_2, B_3, \dots, B_s$, we have $s - 1$ disjoint clique-sum decompositions such that we may view B_i as the root of each of these decompositions for $2 \leq i \leq s$. For each of these decompositions, we do the same thing. In this way, we have finitely many disjoint clique-sum decompositions $Q_1, \dots$ such that each bag of the decomposition has either small order or an h-almost-embeddable structure. In addition, these clique-sum decompositions $Q_1, \dots$ do not have a vertex set W with $|W| \leq k - 2$ such that $W = B'_t \cap B'_s$ for two bags B'_s, B'_t, where B'_s is a parent of B'_t. Furthermore, we can order these decompositions $Q_1, \dots$, such that, if we can list-color Q_1, then we can list-color their children, and so on by using the "precoloring at most $k-2$ vertices" technique (the precoloring comes from the previous list coloring), as discussed in Section 4.

Finding such separations can clearly be done in polynomial time once we are given the clique-sum decomposition.

Step 2. Refinement of the clique-sum decomposition.

At this moment, we have finitely many clique-sum decompositions $Q_1, \dots$, but there is an order for these decompositions. In Step 2, we apply the following algorithm to each of these decompositions with at most $k-2$ precolored vertices in the apex vertex set of the root. The coloring of these precolored vertices will be specified later in this order. First, look at the decomposition Q_1 containing the root B_1 with X' precolored. Refine the clique-sum decomposition as in Theorem 2.1. Do the same thing for each of Q_i. This can be done in time $n^{O(h)}$.

Step 3. List-coloring the clique-sum decomposition.

For each of the decompositions $Q_1, \dots$, we first determine $p_i \leq \chi_l(Q_i)$. The number p_i comes when we apply the bounded-treewidth method. We then select the maximum p_i, and output $p_i + k - 2$. Suppose now that lists L with each vertex having at least $p_i + k - 2$ colors are given. List-color Q_1 first, using the argument in the previous section. Then, in order, we list-color $Q_2, \dots$, with precolored vertices of order at most $k - 2$ in the apex set of the root. The precoloring comes from the previous coloring, and it involves at most $k - 2$ vertices.

All the arguments are constructive in a sense that we can convert them into a polynomial-time algorithm. As shown in the previous section, we can list-color the whole graph using at most $\chi_l(G) + k - 2$ colors. The time complexity is $n^{O(h)}$. This completes the proof.

6 Extension to Graph Coloring

Theorem 1.3 clearly gives rise to an additive approximation algorithm for graph-coloring K_k-minor-free graphs within

$k-2$ of the chromatic number. In the special case of graph coloring, however, we can improve the $k-2$ bound to $k-3$:

THEOREM 6.1. *For any graph G without K_k minors, there is a polynomial-time algorithm to color the graph G using at most $\chi(G)+k-3$ colors.*

The proof is almost identical, except for the following:

1. We are allowed to precolor at most $k-2$ vertices (the vertex set X') in the apex vertex set of the root of the clique-sum decomposition. But when we apply dynamic programming to $N-X$, we put one vertex x in X' to $N-X'$ with x being precolored. We can easily get an optimal coloring of $(N-X')\cup\{x\}$, by possibly recoloring $N-X'$ (but x is still precolored). Therefore, we can get a graph coloring of N using $\chi(G)+k-3$ colors.

2. When we apply the argument in Section 4.2, we can gain one color for graph coloring.

   More precisely, if there are two bags B_j and B_{j+1} such that B_j is a parent of B_{j+1}, and $|B_j\cap B_{j+1}|\leq k-2$, then this means that the clique-sum has a separation (A,B) such that $A\cap B=B_j\cap B_{j+1}$, $|A\cap B|\leq k-2$, and A contains B_j and B contains B_{j+1}. In addition, all the vertices in $B_j\cap B_{j+1}$ are contained in the apex vertex set of B_{j+1}. We can think of B_{j+1} as the root of the clique-sum decomposition of B. In this case, we first apply induction to A, and after graph-coloring A, we apply induction to B with precoloring of either $A\cap B$ (if the vertices in $A\cap B$ use at most $k-3$ colors) or $(A\cap B)-\{v\}$ for some vertex v in $A\cap B$ (else). In either case, the precoloring comes from the coloring of A. In the first case, it is easy to save one color as claimed. Suppose the second case happens. Then v is precolored by the coloring of A, but by possibly recoloring B (with precoloring of $(A\cap B)-\{v\}$), the color of v in B matches the color of v in A, because $A\cap B$ uses exactly $k-2$ colors. Thus we can save "one" color as claimed.

3. Theorem 3.1 still holds if all the vertices that have no neighbors in the cuffs $S_1,S_2,\ldots,S_d$ have at least five available colors in their lists, and each vertex that has a neighbor in $S_1,S_2,\ldots,S_d$ has at least three available colors in its list that are not used in its colored neighbors under the precoloring of $S_1,S_2,\ldots,S_d$. This will be proved somewhere else, but if we just need the usual graph-coloring version, then it would follow from the proof in [33] together with the result in [27]. More precisely, the following is true:

   For any three nonnegative integers g,q,d (with $q\geq 4$), there exists a natural number $r(g,q,d)$ such that the following holds. Suppose that G is embedded on a fixed surface S of Euler genus g and of the representativity at least $r(g,q,d)$, and there are d disjoint cuffs $S_1,S_2,\ldots,S_d$ such that the distance (in the sense of the Robertson-Seymour metric) of any two cuffs of $S_1,S_2,\ldots,S_d$ is at least q. Suppose furthermore that all the vertices in $S_1,S_2,\ldots,S_d$ are precolored using at most $\chi(G)$ colors. Moreover, the following conditions are satisfied:

   (a) all the faces except for $S_1,S_2,\ldots,S_d$ are triangles; and

   (b) no vertex v of $G-(S_1\cup S_2\cup\cdots\cup S_d)$ is joined to more than $\chi(G)-1$ colors unless v has degree 4, or v has degree 5 and v is joined to two vertices of the same color.

   Then the precoloring of the cuffs $S_1,S_2,\ldots,S_d$ using at most $\chi(G)\geq 3$ colors can be extended to a $(\chi(G)+2)$-coloring of G. In fact, in the surface part, we need only five colors for all of the vertices. (Precoloring may use more than five colors and vertices that are adjacent to precolored vertices may need some other color, though.) Also, there is a polynomial-time algorithm for such a coloring G. The same conclusion holds if we replace $\chi(G)$ by any number $p\geq 3$.

For the complete proof, see [21].

The above first and second arguments are not possible for list coloring. Therefore, we have to stick to the additive approximation within $k-2$ of the list chromatic number.

7 Concluding Remarks

We gave an additive approximation algorithm for list-coloring K_k-minor-free graphs within $k-2$ of the list chromatic number. The bound $k-2$ may not be best possible. One natural conjecture is the following.

CONJECTURE 7.1. *There is an additive approximation algorithm for list-coloring H-minor-free graphs within c of the list chromatic number for an absolute constant c.*

Robin Thomas (private communication) has conjectured the graph-coloring case of Conjecture 7.1. This is still open, and as far as we see, the best results known are the 2-approximation algorithm in [6], and Theorem 6.1.

Our approach clearly breaks down, even for the graph-coloring case. Let us highlight some technical difficulties.

1. When there is a separation of order at most $k-2$ in the clique-sum decomposition of two adjacent bags, we have broken into two parts. This allows us to list-color one part first, and then extend the coloring of it to the other part.

   But if the size of the separation is too small, say at most c, which does not depend on k, then the structure as in Theorem 2.1 does not seem to help.

   Let us observe that, if both G and H are K_k-minor-free graphs, then the graph G' obtained from G and H by clique-sum of size at most $k-2$ is K_k-minor-free. Therefore, we need to deal with the clique-sum of size at most $k-2$.

2. If we cannot control clique-sums, then our decomposition approach does not seem feasible.

   In fact, the following does not hold: there exists a vertex partition into two parts V_1, V_2 such that V_1 has bounded treewidth and V_2 has chromatic number at most c for an absolute constant c.

   Robin Thomas and the first author constructed the following example:

   Suppose G has a clique-sum decomposition such that the size of each joint set is exactly $k-2$ (that is, the intersection between any two consecutive pieces has exactly $k-2$ vertices). Suppose furthermore that every piece consists of a planar triangulation with exactly $k-5$ apex vertices (such that these $k-5$ vertices consist of a clique). Moreover, every piece, except for the root, contains exactly three vertices of a face in the parent piece (and these three vertices are in the apex vertex set), and every face of each piece is involved in a clique-sum of its children. So each joint set consists of a face (of three vertices) and exactly $k-5$ vertices in the apex vertex set of each bag. Then this graph is K_k-minor-free, and it may not have a vertex partition into two parts V_1, V_2 such that V_1 has bounded treewidth and V_2 has chromatic number at most $k/3$, because both parts may contain a clique of size at least $k/3$. Therefore, this example kills our approach.

3. Moreover, in order to attack Conjecture 7.1, we may need to figure out the following: Suppose the clique-sum decomposition is given. How many neighbors can each vertex in the surface (of each bag) have in the apex vertex set? In our proof, we can prove that it can have at most $k-5$ neighbors in the apex vertex set of each bag.

So far, we do not see how to overcome any of these issues. It seems that we need a significant generalization of Robertson-Seymour's structure theorem [28], which is presently out of reach for general H-minor-free graphs.

In other work, we have considered the special case of apex-minor-free graphs, where the excluded minor H has the *apex* property that deleting some vertex results in a planar graph. In this case we can establish a stronger form of the Robertson-Seymour decomposition, using techniques very similar to this paper. Specifically, we have proved that the apex vertices can be constrained to have edges only to vertices of vortices, but we have to generalize vortices to what we call "quasivortices", which have bounded treewidth instead of pathwidth. The result is also algorithmic:

THEOREM 7.1. *For any fixed apex graph H, there is a constant h such that any H-minor-free graph can be written as a clique-sum of h-almost-embeddable graphs such that the apex vertices in each piece are only adjacent to quasivortices. Moreover, apices in each piece are not involved in the surface part of other pieces. Furthermore, there is a polynomial-time algorithm to construct this clique-sum decomposition for a given H-minor-free graph.*

Theorem 7.1 has several applications to approximation algorithms. Among them, we can obtain a solution to Conjecture 7.1 for graph coloring in apex-minor-free graphs, specifically an additive 2-approximation:

THEOREM 7.2. *For any apex graph H, there is a polynomial-time additive approximation algorithm that colors any given H-minor-free graph using at most 2 more colors than the optimal chromatic number.*

This additive bound of 2 matches Thomassen's results [33] for bounded-genus graphs, which are a special case of apex-minor-free graphs. This result is essentially best possible: distinguishing between 3 and 4 colorability is NP-complete on any fixed surface, and distinguishing between 4 and 5 colorability would require a significant generalization of the Four Color Theorem characterizing 4-colorability in fixed surfaces.

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A Basic Definitions

In this paper, an *embedding* refers to a 2-*cell embedding*, i.e., a drawing of the vertices and edges of the graph as points and arcs in a surface such that every face (region outlined by edges) is homeomorphic to a disk. A *noose* in such an embedding is a simple closed curve on the surface that meets the graph only at vertices. The *length* of a noose is the number of vertices it visits. The *representativity* or *face-width* of an embedded graph is the length of the shortest noose that cannot be contracted to a point on the surface. The *Robertson-Seymour metric* [26, 27] defines the distance between two *atoms* (vertices, edges, or faces) is half the length of the shortest contractible noose containing the atoms, or the representativity if that is smaller.

A graph G is h-*almost embeddable* in S if there exists a vertex set X of size at most h (called the *apices*) such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$, where

1. G_0 has an embedding in S;
2. the graphs G_i, called *vortices*, are pairwise disjoint;
3. there are faces $F_1, \ldots, F_h$ of G_0 in S, and there are pairwise disjoint disks $D_1, \ldots, D_h$ in S, such that for $i = 1, \ldots, h$, $D_i \subset F_i$ and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap D_i$ (the vertices in U_i are called *society vertices*); and
4. the graph G_i has a path decomposition $(\mathcal{B}_u)_{u \in U_i}$ of width less than h, such that $u \in \mathcal{B}_u$ for all $u \in U_i$. The sets $\mathcal{B}_u$ are ordered by the ordering of their indices u as points along the boundary cycle of face F_i in G_0.

Suppose G_1 and G_2 are graphs with disjoint vertex sets and let $k \geq 0$ be an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G_i' be obtained from G_i by deleting some (possibly no) edges from the induced subgraph $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \rightarrow W_2$. We define a *clique k-sum* G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G_1' and G_2' by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*. Note that each vertex v of G has a corresponding vertex in G_1 or G_2 or both. Also, $\oplus$ is not a well-defined operator: it can have a set of possible results.

See [6, 20, 28] for more definitions.