A Note on Reconfiguring Tree Linkages: Trees can Lock

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Abstract

It has recently been shown that any polygonal chain in the plane can be reconfigured to lie on a straight line, and any polygon can be reconfigured to be convex. This result cannot be extended to tree linkages: we show that there are trees with two configurations that are not connected by a motion. Indeed, we prove that an N-link tree can have $2^{\Omega(N)}$ equivalence classes of configurations.

Key words: Motion Planning, Linkage Reconfiguration, Graph Embedding, Tree Embedding, Distance Geometry.

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1 Introduction

A *polygonal chain* is a connected series of line segments in the plane. Chains may be open, or closed to form a polygon. A *simple* chain is one that does not self-intersect: only segments adjacent in the chain intersect, and then only at their shared endpoint. If the segments of a polygonal chain are viewed as rigid bars, and the vertices as universal joints, natural questions are whether every open chain can be *straightened*, i.e., reconfigured to lie on a straight line, and whether every closed chain can be *convexified*, i.e., reconfigured to form a planar convex polygon. In both cases, the chains are to remain simple throughout the motion. If a chain cannot be so reconfigured, it is called *locked*. Connelly, Demaine, and Rote [2] have recently shown that no chain or polygon is locked; Streinu [3] provides an alternative motion.

There is a natural equivalence relation on the set of configurations of a linkage: two configurations are equivalent if there is a motion that takes the linkage from one configuration to the other. The Connelly-Demaine-Rote result states that each open chain has a single equivalence class of configurations. In this note we show that their result cannot be generalized to trees: a tree linkage can have many configuration classes.

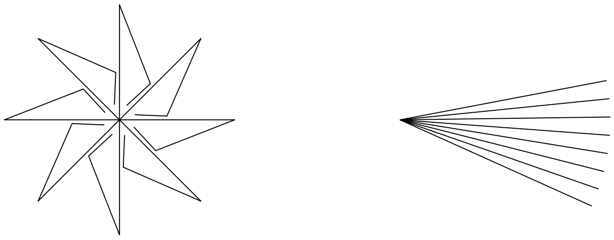

Fig. 1. A tree linkage with disconnected configuration space: no motion is possible from the one configuration to the other.

The trees considered in this paper consist of n *petals*, each comprised of three links. Two configurations of such a tree are sketched in Figure 1. The petals

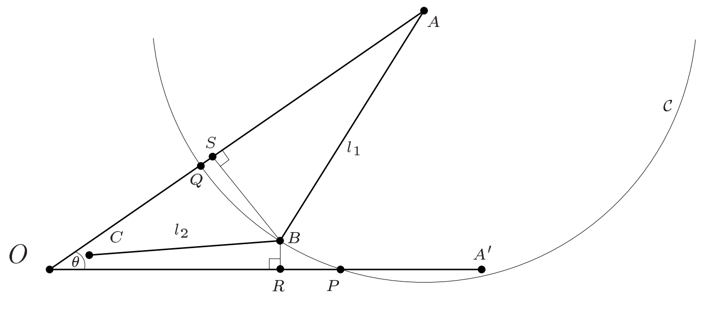

Fig. 2. A petal $OABC$ of the tree.

all meet at central joint O and the other joints of each petal are labelled A, B, and C; see Figure 2. The joint corresponding to A in the clockwise neighbouring petal is denoted A', and $\theta \equiv \angle A'OA$ is called the *petal angle*. Corresponding link lengths are the same in all petals: $\|OA\| = 1$, $\|AB\| \equiv l_1$, and $\|BC\| \equiv l_2$.

Our construction of a locked tree requires choosing the parameters n, l_1, and l_2 to satisfy two sets of conditions. The first set of conditions ensures that it is possible to initially place the tree in a configuration, sketched on the left of Figure 1, in which each petal configuration is of a "restricted" type described in Section 2. In Section 3 we introduce the second set of conditions, which ensure that for any motion starting from such a configuration, each petal angle is bounded away from zero, proving that the configuration sketched on the right of Figure 2 is in a different configuration class. The tree is locked, in essence, because the lower bound on the petal angles is such that squeezing $n-1$ petals does not leave enough angle for the last petal to unfold.

In what follows, we choose $n = 5$, $l_1 = 0.9511$, and $l_2 = 0.299$ for the verification that the two sets of conditions can be satisfied (in fact, a similar construction is possible for any $n > 4$ [1]).

2 Restricted Petal Configuration

We need the following technical definition. For any configuration of a petal, let $\mathcal{C}$ be the circle centred at A with radius $l_1 < 1$. Let β be the petal angle at which $\mathcal{C}$ is tangent to OA', i.e. $\beta \equiv \arcsin l_1 \approx 1.25678$. When $\theta < \beta$, circle $\mathcal{C}$ will properly intersect OA'; let P be the intersection point nearest O. Circle $\mathcal{C}$ will also intersect OA and we denote this point by Q. If B lies on the small arc PQ of $\mathcal{C}$, then B lies inside $\triangle A'OA$. Drop perpendiculars from B to legs OA' and OA. Let $R \in OA'$ be such that $BR \perp OA'$ and $S \in OA$ be such that $BS \perp OA$.

A petal configuration is said to be *restricted* if the following conditions hold: (i) $\theta < \beta$, (ii) B is on the open small arc PQ of $\mathcal{C}$, and (iii) C is in the open region bounded by the quadrilateral $\square ORBS$. The petal in Figure 2 is in a restricted configuration.

Let us verify that our tree may be initially configured with each petal in a restricted configuration, and with equal petal angles. Condition (i) is satisfied, as $\theta = 2\pi/5 \approx 1.25664$. Joints B and C can be placed appropriately, since $\|AB\| < 1$, and $\|BC\| = 0.2990$ is less than $\|OP\| \approx 0.2999$.

3 Locking the Tree

We wish to ensure that if a petal begins in a restricted configuration and the petal angle remains small, that joint C remains trapped in $\square ORBS$. Since $l_2 > \sin\beta\cos\beta$, it follows that in a restricted configuration, both $\|BR\|$ and $\|BS\|$ are strictly less than l_2. We use this to prove the next lemma.

Lemma 1 *Consider a petal initially in a restricted configuration. Throughout any motion during which θ is strictly less than β, the petal remains in a restricted configuration.*

Proof: Consider, in turn, the three conditions required for a restricted configuration. Condition (i) holds throughout the motion by assumption. As $\theta < \beta$, $\mathcal{C}$ properly intersects links OA' and OA so condition (ii) holds. As C starts the motion inside $\square ORBS$, condition (iii) is violated only if C passes through one of the (non-link) sides of this quadrilateral. By the remark above, $\|BR\|$ and $\|BS\|$ are both strictly less than $\|BC\|$, so C cannot pass through side BR or side BS. Thus, condition (iii) holds throughout the motion. $\square$

Consider the non-simple configuration in which joint C coincides with joint O, and joint B lies on OA'. The resulting petal angle is $\theta_m \approx 1.25661$. Simple trigonometry shows that in a restricted configuration, $\theta \geq \theta_m$.

A final condition is required to make the tree locked. Consider a configuration in which some petal angle is greater than or equal to β. It follows that some other petal angle must be less than or equal to $\alpha \equiv (2\pi - \beta)/(n-1) \approx 1.25660$. Notice that $\alpha < \theta_m$; this fact will ultimately make reaching such a configuration impossible, as the proof of Lemma 2 below will show.

4 Discussion and Corollary

The initial tree configuration has all petals in a restricted configuration with each petal angle in the open interval (α, β). The next lemma shows that no motion exists to take the linkage to the configuration sketched in the right hand side of Figure 1, in which at least one petal angle is less than α.

Lemma 2 *Consider a tree linkage with each petal in a restricted configuration. During any motion, each petal angle remains in the range (α, β).*

Proof: Suppose, to the contrary, a motion exists that takes some petal angle out of the range (α, β). No petal angle may reach β before some petal angle reaches α. Consider the first petal to reach angle α. This petal angle has never exceeded β, so Lemma 1 ensures that the petal configuration remains restricted. This means that the petal angle is bounded from below by θ_m, contradicting the fact that the petal angle reached $\alpha < \theta_m$. □

Our tree, therefore, has at least two equivalence classes of configurations.

The construction given can be extended to yield an N-link tree with $2^{\Omega(N)}$ equivalence classes of configurations. Consider the linkage in Figure 3, in

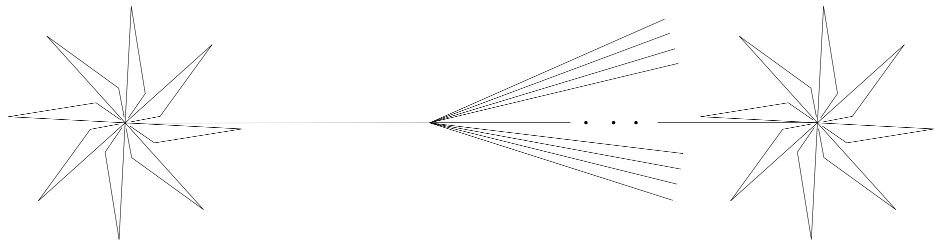

Fig. 3. A tree linkage formed by joining k copies of a lockable tree.

which there are k copies of our tree connected by long links joining the O joints of the subtrees. Each subtree has at least two distinct configuration classes, so the tree as a whole has at least $2^k \in 2^{\Omega(N)}$ configuration classes.

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