On Reconfiguring Tree Linkages: Trees can Lock

Therese Biedl* Erik Demaine* Martin Demaine*
Sylvain Lazard† Anna Lubiw* Joseph O'Rourke‡ Steve Robbins§
Ileana Streinu‡ Godfried Toussaint§ Sue Whitesides§

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Abstract

It has recently been shown that any simple (i.e. nonintersecting) polygonal chain in the plane can be reconfigured to lie on a straight line, and any simple polygon can be reconfigured to be convex. This result cannot be extended to tree linkages: we show that there are trees with two simple configurations that are not connected by a motion that preserves simplicity throughout the motion. Indeed, we prove that an N-link tree can have $2^{\Omega(N)}$ equivalence classes of configurations.

1 Introduction

Consider a graph, each edge labelled with a positive number. Such a graph may be thought of as a collection of distance constraints between pairs of points in a Euclidean space. A *realization* of such a graph maps each vertex to a point, also called a *joint*, and maps each edge to the closed line segment, called a *link*, connecting its incident joints. The link length must equal the label of the underlying graph edge. If a graph has one or more such realizations, we call it a *linkage*.

An embedding of a linkage in space is called a *configuration* of the linkage if any pair of links whose underlying edges are incident on a common vertex intersect only at the common joint and all other pairs of links are disjoint. Some authors allow the term configuration to refer to objects that self-intersect. In

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*University of Waterloo, Waterloo, Canada. {biedl, eddemaine, mldemaine, alubiw}@uwaterloo.ca.

†INRIA Lorraine, France. lazard@loria.fr.

‡Smith College, Northampton, USA. {orourke, streinu}@cs.smith.edu.

§McGill University, Montreal, Canada. {stever, godfried, sue}@cgm.cs.mcgill.ca.

contrast, we require all configurations to be simple; i.e. non self-intersecting. A *motion* of a linkage is a continuous movement of its joints such that it remains in a valid configuration at all times. A natural question is whether a motion exists between two given configurations of a linkage.

For a linkage in the plane whose underlying graph is a path, a related question is whether it can always be *straightened*; i.e. whether it can be moved from any configuration to lie on a straight line. Similarly, we wonder whether a cycle linkage (polygon) can always be *convexified*; i.e. whether it can be moved to a configuration that is a convex polygon. If a linkage cannot be so reconfigured, it is called *locked*. These questions have been in the math community since the 1970's [22] and in the computational geometry community since 1991 [18, 19], but first appeared in print in 1993 and 1995: [20] and [16, p. 270]. Initial computational geometry results focused on certain classes of configurations such as "visible" chains [4], star-shaped polygons [9] and monotone polygons [3]. Connelly, Demaine, and Rote have recently proved that in the plane, no chain or polygon is locked [7]; Streinu [28] provides an alternative proof. In three dimensions, while a complete characterization isn't known, there are configurations of open polygonal chains and of polygons that can be straightened, or convexified, respectively, and other configurations that can not be [1]. In four or more dimensions, no chain or polygon is

Related linkage motion results in the computational geometry literature (e.g. [10–14, 17, 21, 23–26, 29–31]) allow the links to cross or to pass through or over one another. In other words, the links represent distance constraints between joints, not physical obstacles that must avoid each other. There is also the very general algebraic approach to motion planning of [5] and [27], since the constraint that the links not cross can be specified algebraically. For related work from a topological point of view, see [15] and references therein; again, this work allows links to cross.

There is a natural equivalence relation on the set of linkage configurations: two configurations are equivalent if there is a motion that takes the linkage from one configuration to the other. The Connelly-Demaine-Rote result states that chains in the plane have a single equivalence class of configurations. In this report we show that their result cannot be generalized to tree linkages: trees can have many configuration classes.

This report establishes that a suitably-constructed tree linkage has two configurations (pictured in Figure 1) for which no motion between them is possible. (This result also answers a question posed in [8], arising from a paper folding problem.) As a corollary, we obtain the result that an N-link tree linkage can exhibit $2^{\Omega(N)}$ equivalence classes of configurations.

The rest of this report is organized as follows. Section 2 gives definitions and the basic idea for constructing a locked tree configuration, Section 3 gives the construction itself, and Sections 4 and 5 give the correctness proof. Section 6 concludes with some open problems.

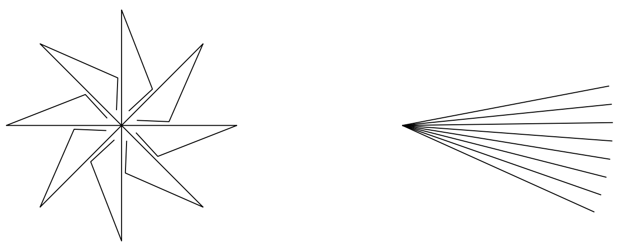

Figure 1: A tree linkage with disconnected configuration space: no motion is possible from one configuration to the other.

2 Preliminaries

In this section, we introduce the technical definitions used. From now on, we often do not distinguish between vertices and edges in the underlying tree, the corresponding joints and links in the tree linkage, and the points and line segments occupied by the joints in links in a particular configuration. The context should make the meaning clear.

The trees considered in this report consist of n *petals*, each comprised of three links. The initial configuration is sketched in Figure 2 and detailed as the report unfolds.

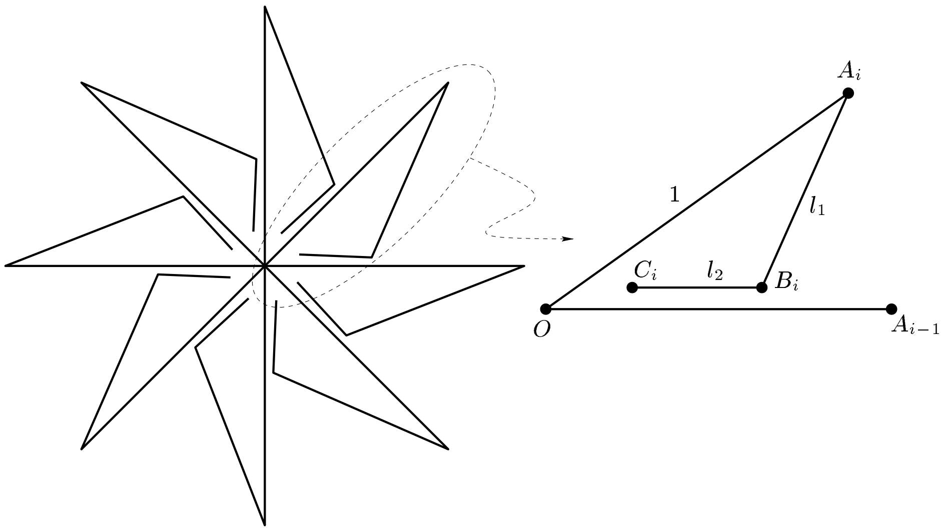

Figure 2: An example linkage with enlarged view of petal i, showing the joint labels and the link lengths.

The petals all meet at joint O; the other joints of petal i are labelled A_i, B_i, and C_i. Designate the *petal angle* as $\theta_i = \angle A_{i-1}OA_i$; let $\bar{\theta} = 2\pi/n$. All index arithmetic is taken mod n; all angles are measured in the interval $[0, 2\pi)$. The link lengths are as follows: $\|OA_i\| = 1$, $\|A_iB_i\| = l_1$, and $\|B_iC_i\| = l_2$, where for two points in the plane, X and Y, we use XY to designate the closed line segment between X and Y, and $\|XY\|$ to denote its length. The values of l_1 and l_2 will be discussed in Section 3.

We often focus on a single petal at a time, so the notation is simplified by suppressing the petal index. Joints of petal i are referred to as A, B, and C, joint A_{i-1} is denoted A', and the petal angle, i.e., $\angle A'OA$, as just θ. Let L be the line through OA', and choose a reference frame with L oriented horizontally, O to the left of A'.

2.1 The Intuition

We first give a brief outline of the main argument, before delving into the details.

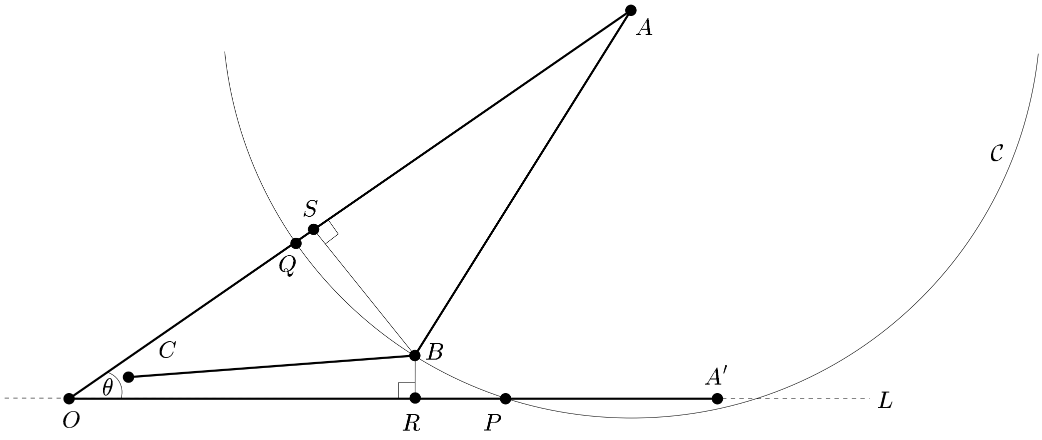

Figure 3: Illustrating the definition of *restricted configuration*.

In the initial configuration, all petals are in a congruent configuration, pictured in Figure 3, with petal angle $\bar{\theta} = 2\pi/n$. By choosing l_1 long enough, we ensure that link AB cannot swing out so as to straighten joint A. Thus, the petal angle must be increased before joint A can straighten. Because the petals all join at O, opening up one petal necessitates squeezing the other petals.

Consider now how small we can squeeze a petal angle. By choosing l_2 long enough, we ensure that link BC cannot swing over to fold up against link AB. In fact, joint C is trapped inside quadrilateral $\square ORBS$. We show later that the smallest petal angle is obtained by moving joint C to O, pictured in Figure 4. Lengthening l_2 increases this minimum petal angle, so by choosing l_2 long enough, we can ensure that squeezing $n-1$ petals to their minimum angle is still not enough to let the last petal open. This, in essence, is why the tree is locked.

From the above discussion, we note that l_1 and l_2 must be chosen "long enough". On the other hand, they must also be "short enough" that the configuration shown in Figure 3 is achievable. Section 3 details all the constraints needed to satisfy these two requirements. Section 4 defines a "restricted" class of petal configurations (to which the configuration in Figure 3 belongs) and proves that there is a non-zero minimum petal angle for this class of configurations. Finally, Section 5 proves that a linkage with parameters n, l_1, and l_2, satisfying all the constraints of Section 3, can be put into a initial configuration that is locked.

3 Constructing a Locked Tree

This section details the constraints on the parameters n, l_1, and l_2 that are necessary to construct our locked linkage. Lemma 1, at the end of this section, demonstrates that the constraints are simultaneously satisfiable. Refer to Figure 3 throughout this section.

A three-link petal cannot be locked if the petal angle is greater than or equal to $\pi/2$. Thus we must have the initial petal angle ($\bar{\theta} = 2\pi/n$) strictly less than $\pi/2$, or

$$n > 4. \tag{1}$$

Henceforth, assume $\theta < \pi/2$. We want to have joint B to the left of the vertical line through A. In order that link AB fits, we must have

$$l_1 < 1. \tag{2}$$

For any configuration of a petal, let $\mathcal{C}$ be the circle centred at A with radius l_1. Let β be the petal angle at which $\mathcal{C}$ is tangent to L,

$$\beta = \arcsin l_1. \tag{3}$$

The range for the principal value of arcsine is $[-\pi/2, \pi/2]$. However, we know that l_1 must be positive, and less than 1 (Inequality 2), so β is actually in the range $(0, \pi/2)$.

When $\theta < \beta$, circle $\mathcal{C}$ will properly intersect L. We want this to be true for the initial configuration, in which all petal angles are $\bar{\theta}$, so we require

$$\bar{\theta} < \beta. \tag{4}$$

Suppose $\mathcal{C}$ properly intersects L and let P be the leftmost intersection point. Applying the cosine rule to $\triangle OAP$ and noting that $\|OA\| = 1$ and $\|AP\| = l_1$, yields $l_1^2 = \|OP\|^2 + 1 - 2\|OP\| \cos\theta$. Since l_1, the radius of $\mathcal{C}$, is strictly less than $\|OA\| = 1$ by Inequality 2, joint O is outside the circle $\mathcal{C}$. From this, and $\theta < \pi/2$, it follows that P is to the right of O, and hence $\|OP\|$ is the smaller root of the quadratic equation. We define the function

$$l_P(\theta) = \|OP\| = \cos\theta - \sqrt{l_1^2 - \sin^2\theta}. \tag{5}$$

This function is only defined on values of θ for which $\mathcal{C}$ intersects L; i.e., for θ in $[0, \beta]$. Differentiating 5 shows that l_P is a strictly increasing function on this interval.

Furthermore, P is to the left of the vertical line through A, and hence $P \in OA'$, but P is not at A'. We know that A is inside the circle $\mathcal{C}$, while O is outside, so $\mathcal{C}$ intersects OA; let Q be this intersection point. Note that the small arc PQ of $\mathcal{C}$ lies inside $\triangle A'OA$.

Suppose $\theta < \beta$, and B is on the small arc PQ of $\mathcal{C}$. Then B is inside $\triangle A'OA$. Since the triangle is isosceles, and the angle θ is $< \pi/2$, $\triangle A'OA$ is acute. This ensures that there is a line passing through B perpendicular to each edge of the triangle. Let $R \in OA'$ be such that $BR \perp OA'$ and $S \in OA$ be such that $BS \perp OA$. We want to have joint C inside quadrilateral $\square ORBS$. This is feasible for the initial configuration if we choose

$$l_2 < l_P(\bar{\theta}), \tag{6}$$

for then we may place B near P, and C along OB. At the same time, we wish to have l_2 long enough that C remains trapped in $\square ORBS$. This is ensured by choosing

$$l_2 > \sin\beta\cos\beta, \tag{7}$$

as we show later in Section 4.

Define

$$\alpha = (2\pi - \beta)/(n-1). \tag{8}$$

Later, we will see that if a petal angle is opened past β, then some other petal angle must be smaller than α. The proof that the tree is locked then hinges on showing that there is a minimal petal angle, which is greater than α.

To obtain a non-zero minimal petal angle, we require

$$l_1 + l_2 > 1, \tag{9}$$

for otherwise, links AB and BC could fold flat against link OA, and the petal angle could squeeze to zero. Indeed, we show later, in Lemma 3, that the minimum possible petal angle is bounded from below by the petal angle obtained in the non-simple configuration with C at O, and $B \in OA'$, pictured in Figure 4. Define θ_m to be the resulting petal angle. With $\triangle OAB$, the cosine rule yields $l_1^2 = l_2^2 + 1 - 2l_2\cos\theta_m$, or

$$\theta_m = \arccos\left(\frac{1 - l_1^2 + l_2^2}{2l_2}\right). \tag{10}$$

We are using the principal value of arccosine, so $\theta_m \in [0, \pi]$.

Finally, in order to prove Theorem 5, we assume that l_1 and l_2 are such that

$$\alpha < \theta_m. \tag{11}$$

We prove, in Appendix A, the following lemma which states that all the constraints of this section may be simultaneously satisfied by an appropriate

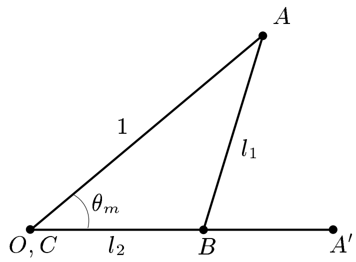

Figure 4: For calculating θ_m.

choice of n, l_1, and l_2. For $n = 5$, calculation shows that $l_1 = 0.9511$ and $l_2 = 0.299$ satisfy the system of inequalities above, with $71.997984 < 71.998224 < 72 < 72.008064$ (angles in degrees) for α, θ_m, $\bar{\theta}$, and β, respectively.

Lemma 1 *For each integer $n > 4$, there exists two real numbers l_1 and l_2 satisfying simultaneously all the constraints of Section 3 (i.e., Inequalities 2, 4, 6, 7, 9, and 11).*

In the sequel, we assume some choice of parameters has been made such that all the constraints of this section hold.

4 Restricted Configurations

Referring to Figure 3, recall from the definition of β (Equation 3) that $\theta < \beta$ implies that circle $\mathcal{C}$ properly intersects link OA'. Thus, points P and Q are well-defined. When joint B is on the small arc PQ, recall that points R and S are also well-defined. And finally, joint C also fits inside $\Box ORBS$, due to Inequality 6.

Definition: A petal configuration is said to be *restricted* if the following conditions hold:

(i) $\theta < \beta$,

(ii) B is on the open small arc PQ of $\mathcal{C}$, and

(iii) C is in the open region bounded by the quadrilateral $\Box ORBS$.

Note in Figure 3 that $\|BR\|$ and $\|BS\|$ are both smaller than $\|BC\|$. We show now that this is always the case in a restricted configuration.

Lemma 2 *In a restricted configuration, both $\|BR\|$ and $\|BS\|$ are strictly less than l_2.*

Proof: For a point X, let $d(X, L)$ denote the distance from X to the line L through O and A'. We have that $\|BR\| = d(B, L)$. Because B is on the small arc PQ, $d(B, L) \leq d(Q, L) = \|OQ\| \sin\theta$. Hence,

$$\|BR\| \leq \|OQ\| \sin\theta. \tag{12}$$

By similar reasoning,

$$\|BS\| \leq \|OP\| \sin\theta. \tag{13}$$

Note that $\|OQ\| = 1 - l_1 = 1 - \sin\beta$, by Equation 3. For $x \in [0, \pi/2]$, $\sin x + \cos x \geq 1$, so $\|OQ\| \leq \cos\beta$. Because l_P is an increasing function, $\|OP\| = l_P(\theta) \leq l_P(\beta)$, and $l_P(\beta) \leq \cos\beta$, by Equation 5.

Both $\|OP\|$ and $\|OQ\|$ are $\leq \cos\beta$. Hence, by Inequalities 12 and 13, both $\|BR\|$ and $\|BS\|$ are bounded from above by $\cos\beta \sin\theta < \cos\beta \sin\beta$, as $\theta < \beta \leq \pi/2$ by assumption. But this is strictly less than l_2 by Inequality 7. □

Lemma 3 *In a restricted configuration,* $\theta \geq \theta_m$.

Proof: Suppose, for contradiction, the petal is in a restricted configuration with $\theta < \theta_m$.

Consider the two triangles $\triangle OAB$ and $\triangle OAP$. These triangles share the common side OA, and $\|AB\| = \|AP\| = l_1$. Two sides of $\triangle OAB$ are equal length with two sides of $\triangle OAP$. Moreover the included angles satisfy $\angle OAB < \angle OAP$, since by definition of restricted configuration (condition (ii)), joint B lies on the small arc PQ of $\mathcal{C}$. Applying the cosine law to the remaining side in each triangle (or using Euclid's Proposition 24, Book I) we see $\angle OAB < \angle OAP$ implies $\|OB\| < \|OP\|$. By definition (Equation 5) $\|OP\| = l_P(\theta)$, which is less than $l_P(\theta_m)$ since l_P is an increasing function. A direct computation shows that $l_P(\theta_m) = l_2$, so we have $\|OB\| < l_2$.

By Lemma 2, we have also $\|BR\| < l_2$ and $\|BS\| < l_2$.

All four points O, R, B and S are strictly inside the circle of radius l_2 centred at B. Joint C is of course on this circle, so it cannot be inside $\Box ORBS$. This contradicts condition (iii) of a restricted configuration. □

Lemma 4 *Consider a petal in a restricted configuration. Throughout any motion during which* θ *is strictly less than* β*, the petal remains in a restricted configuration.*

Proof: Note that the points P, Q, R, S, and the circle $\mathcal{C}$, are defined by the positions of the joints A and B; as the joints move, so do the points P, Q, etc. For simplicity, we omit displaying this dependence on time.

Consider, in turn, the three conditions required for a restricted configuration. Condition (i) holds throughout the motion by assumption.

In any configuration, B must be on $\mathcal{C}$. Since B starts the motion on the small arc PQ, for condition (ii) to be violated, B must pass through point P or through point Q. P and Q are on the interior of links OA' and OA, respectively, and since $\theta < \beta$, $\mathcal{C}$ always properly intersects these links. Thus B may not move through P or Q, and hence condition (ii) holds throughout the motion.

Given that B remains on the small arc PQ, points R and S are well defined. As C starts the motion inside $\square ORBS$, condition (iii) is violated only if C passes through one of the sides of this quadrilateral. Sides OR and OS are portions of links, so C may not pass through them. By Lemma 2, $\|BR\|$ and $\|BS\|$ are both strictly less than $\|BC\|$, so C cannot pass through side BR or side BS. Thus, condition (iii) holds throughout the motion. $\square$

5 Trees Can Lock

This section describes our main result: two inequivalent configurations of a tree linkage.

Recall that θ_m is defined by a triangle with sides 1, l_1, and l_2, pictured in Figure 4. With $\triangle OAB$, the cosine rule yields

$$l_1^2 = l_2^2 + 1 - 2l_2 \cos\theta_m,$$

while from Inequality 6 we can obtain

$$l_1^2 < l_2^2 + 1 - 2l_2 \cos\bar{\theta},$$

which implies $\theta_m < \bar{\theta}$. Putting this together with Inequalities 11 and 4, we have

$$\alpha < \theta_m < \bar{\theta} < \beta, \tag{14}$$

hence (α, β) is a non-empty interval.

Consider a tree in a configuration in which the petal configurations are all congruent, with petal angles all equal to $\bar{\theta} \in (\alpha, \beta)$. We place joint B on the small arc PQ. Because $l_2 < l_P(\bar{\theta}) = \|OP\|$ (Inequality 6) we can place B near, but not at, P to ensure $\|OB\| > \|BC\|$. This ensures that joint C may be placed along OB, which is inside $\square ORBS$. This is a valid configuration, and furthermore each petal configuration restricted. The next theorem shows all petals remain bounded from below by α.

Theorem 5 *Consider a tree of n petals in a configuration such that $\theta_i \in (\alpha, \beta)$ for $0 \le i < n$, and with each petal in a restricted configuration. During any motion, all petal angles remain in the range (α, β).*

Proof: Suppose, to the contrary, a motion exists that takes some petal angle out of the range (α, β). Let t_α be the first instant that some petal angle, say for petal k, reaches α. Let t_β be the first instant that some petal angle reaches β.

If $t_\beta < t_\alpha$, then at time t_β all angles are strictly greater than α and at least one is equal to β. This means

$$2\pi = \sum_{i=0}^{n-1} \theta_i > (n-1)\alpha + \beta = 2\pi,$$

a contradiction. Hence $t_\alpha \leq t_\beta$.

During the supposed motion, the joint angles change continuously in time. Since $\alpha < \theta_m$ by Inequality 11, and θ_k approaches α from above as t approaches t_α from below, we may choose $t_0 < t_\alpha$ such that $\theta_k \in (\alpha, \theta_m)$ at time t_0.

Note that during the motion up to time t_0, all petal angles are strictly less than β, as $t_0 < t_\alpha \leq t_\beta$. By Lemma 4 all petals remain in a restricted configuration before time t_0, so Lemma 3 applies to petal k. This means $\theta_k \geq \theta_m$, contradicting the choice of t_0. □

Recall that two simple configurations of a tree linkage are equivalent if one can be moved to the other. Our main result is that a tree linkage can have two inequivalent configurations: one with all petals in a restricted configuration, and the other with one or more petal angles less than α. These configurations are illustrated in Figure 1. This result can easily be extended to the following corollary.

Corollary 6 *There exist N-link tree linkages such that the linkages have $2^{\Omega(N)}$ equivalence classes of simple configurations.*

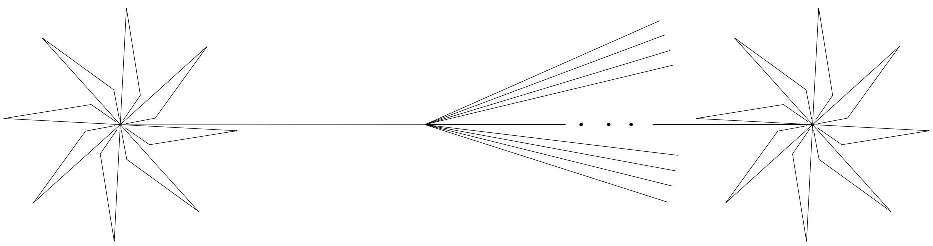

Figure 5: A tree linkage formed by joining k copies of a lockable tree. Each subtree may be in either an open (as is the middle subtree) or a closed (the first and last subtrees) configuration. This linkage has at least 2^k configuration classes.

Proof: Consider the linkage in Figure 5, in which there are k copies of an eight-petal lockable tree connected by long links joining the O joints of the subtrees. The connecting links are long enough that when they are stretched out to form a straight chain, each subtree can be in either an open or a closed configuration without crossing links.

Consider simple configurations in which the long links form a straight chain, and each subtree is in either an open or a closed configuration. Label such configurations by a k-bit vector, specifying for each subtree, whether its configuration is open or closed. Configurations with different labels are clearly not equivalent, as a motion of the entire linkage that would take some subtree in a closed configuration to an open configuration would imply, by removing links outside the subtree, the existence of a motion that would make petal angle of the subtree inferior to α. Hence the number of inequivalent configurations is at least $2^k \in 2^{\Omega(N)}$, as $N = 24k + (k-1)$. $\square$

6 Conclusion

While no chain or polygon in the plane may lock, we showed in this report that for a tree linkage can; i.e. that there can be more than one equivalence class of simple configurations. Indeed, some N-link trees have $2^{\Omega(N)}$ equivalence classes.

The tree construction of Section 3 constrains the link lengths to be non-equal (it appears difficult to even get them nearly equal). This prompts the following question: can a tree linkage with equal-length links have a locked configuration? One "nearly equilateral" tree linkage is shown in Figure 6. We conjecture that if the link lengths are very nearly equal, this configuration is locked; the intuition is as follows. Each of the six triangular petals cannot collapse, so each remains nearly an equilateral triangle. If that is the case, it seems that each petal can only move by pivoting about its degree-3 joint, which it cannot do without crossing a link of the adjacent petal. Notice, however, that if the links truly

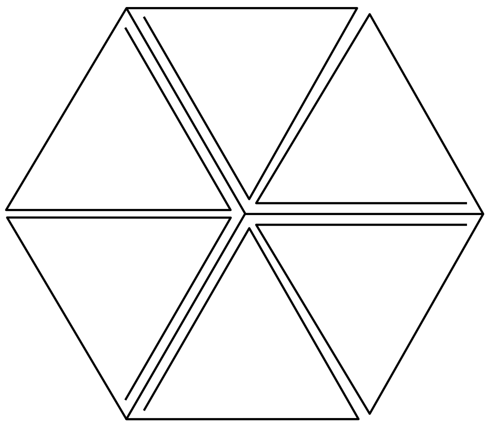

Figure 6: A nearly equilateral tree linkage configuration.

are of equal length, the configuration pictured cannot be simple.

The nearly equilateral example has the feature that the graph has maximum degree three. It is easy to replace a high-degree joint with a number of degree-three joints joined by a chain of tiny links. Do equilateral locked tree linkages with maximum degree three exist?

Finally, many interesting questions can be posed for linkages moving in higher dimensions. See [1, 6] for recent work on chain and cycle linkages moving under simple motion in three and more dimensions.

Acknowledgments

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Our locked tree was inspired by a polygonal chain designed by Joe Mitchell, kindly shared with one of the coauthors. In particular, we borrowed the circular structure from his example.

A preliminary version of this work appeared in [2].

A Proof of Lemma 1

This appendix proves the following lemma.

> For each integer $n > 4$, there exists two real numbers l_1 and l_2 satisfying simultaneously all the constraints of Section 3 (i.e., Inequalities 2, 4, 6, 7, 9, and 11).

Proof:

We show that $l_1 = \sin(\bar{\theta} + \epsilon)$ and $l_2 = l_P(\bar{\theta} - \epsilon/n)$ are feasible link lengths, where $\epsilon = 0.01°$ is feasible for $n = 5, n = 6$ and any $0 < \epsilon < 0.4°$ is feasible for $n \geq 7$. The proof consists of checking, in turn, each of the constraints mentioned above.

Constraint 2: $l_1 < 1$

Given our choice of l_1, this is satisfied as long as $\bar{\theta} + \epsilon < \pi/2$. Recall that $\bar{\theta} = 2\pi/n$ and $n \geq 5$ (Inequality 1), so $\bar{\theta} \leq \frac{2\pi}{5}$. Since $\epsilon < 0.4°$, the constraint is satisfied.

Constraint 4: $\bar{\theta} < \beta$

By the definition of β (Equation 3), and our choice of l_1, we have $\beta = \bar{\theta} + \epsilon$. Since $\epsilon > 0$, the constraint is satisfied.

Constraint 6: $l_2 < l_P(\bar{\theta})$

Since $\epsilon > 0$, $\bar{\theta} - \epsilon/n < \bar{\theta}$. Using the definition $\bar{\theta} = 2\pi/n$, we see that $\bar{\theta} - \epsilon/n = (2\pi - \epsilon)/n$. Since $\epsilon < 0.4°$, $\bar{\theta} - \epsilon/n > 0$. Now, $l_P(\theta)$ is an increasing function on $[0, \beta] \supset (0, \bar{\theta})$, and $l_2 = l_P(\bar{\theta} - \epsilon/n)$, so the constraint is satisfied.

We summarize, for future reference, some inequalities derived so far,

$$0 < \bar{\theta} - \epsilon/n < \bar{\theta} < \beta = \bar{\theta} + \epsilon < \pi/2. \tag{15}$$

Constraint 7: $l_2 > \sin\beta\cos\beta$

This constraint is the only one for which we distinguish cases based on n. For $n = 5$ and $n = 6$, a direct calculation shows that the given link lengths (with $\epsilon = 0.01°$) satisfy the constraint.

For the case $n \geq 7$, we show that any $0 < \epsilon < 0.4°$ yields $l_2 > \frac{1}{2}$, whence the constraint follows since $\sin\beta\cos\beta = \frac{1}{2}\sin(2\beta) \leq \frac{1}{2}$ (the equality is an identity, the inequality uses $\sin x \leq 1$).

Plugging our choice of l_2 into the definition of l_P (5),

$$l_2 = l_P(\bar{\theta} - \epsilon/n) = \cos(\bar{\theta} - \epsilon/n) - \sqrt{l_1^2 - \sin^2(\bar{\theta} - \epsilon/n)}. \tag{16}$$

Cosine is a decreasing function on $[0, \pi]$, so $\cos(\bar{\theta} - \epsilon/n) > \cos\bar{\theta} > \cos(\frac{2\pi}{7})$ given $0 < \bar{\theta} - \epsilon/n < \bar{\theta} < 2\pi/7$ (15 and $n \geq 7$).

Under the radical of 16, substituting $l_1 = \sin(\bar{\theta} + \epsilon)$ gives the expression $\sin^2(\bar{\theta}+\epsilon) - \sin^2(\bar{\theta}-\epsilon/n)$. Using the identity $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$, this expression becomes $\sin(2\bar{\theta} + \epsilon - \epsilon/n)\sin(\epsilon + \epsilon/n) \leq \sin(\epsilon + \epsilon/n) < \sin(2\epsilon) < \sin(0.8°)$. The last two inequalities follow by noting that $0 < \epsilon + \epsilon/n < 2\epsilon < 0.8° < \pi/2$, and sine is increasing on the interval $[0, \pi/2]$.

Using these bounds for the two terms in Equation 16,

$$l_2 > \cos\left(\frac{2\pi}{7}\right) - \sqrt{\sin(0.8°)} \approx 0.505 > \frac{1}{2}.$$

Constraint 9: $l_1 + l_2 > 1$

Define function $f(\theta) = l_1 + l_P(\theta)$ on $[0, \beta]$. We know l_P is a strictly increasing function on $[0, \beta]$, hence so is f. Furthermore, $f(0) = 1$, so $f(\theta) > 1$ for $\theta \in (0, \beta]$.

From 15, we see $\bar{\theta} - \epsilon/n \in (0, \beta)$, so $f(\bar{\theta} - \epsilon/n) > 1$. By definition of f, this becomes $l_1 + l_P(\bar{\theta} - \epsilon/n) > 1$. Since $l_2 = l_P(\bar{\theta} - \epsilon/n)$, we obtain that $l_1 + l_2 > 1$ as desired.

Constraint 11: $\alpha < \theta_m$

By definition, $\alpha = (2\pi - \beta)/(n-1)$ (Equation 8). This shows $\alpha \geq 0$. Rewriting 2π as $n\bar{\theta}$, and β as $\bar{\theta} + \epsilon$ (15), the equation for α becomes $\alpha = (n\bar{\theta} - \bar{\theta} - \epsilon)/(n-1) = \bar{\theta} - \epsilon/(n-1) < \bar{\theta} - \epsilon/n$. Combining all this with (15), we find

$$0 < \alpha < \bar{\theta} - \epsilon/n < \bar{\theta} < \beta < \pi/2. \tag{17}$$

From (17) we see that $\alpha \in (0, \pi)$. By definition (Equation 10), θ_m is also in $(0, \pi)$. Since cosine is decreasing on this interval, the constraint $\alpha < \theta_m$ holds if, and only if, $\cos\alpha > \cos\theta_m$. Using the definition of θ_m (Equation 10), this latter inequality becomes $2l_2\cos\alpha > 1 - l_1^2 + l_2^2$. We collect the terms in l_2 to one side, $l_2^2 - 2l_2\cos\alpha < l_1^2 - 1$, and complete the square to get

$$(l_2 - \cos\alpha)^2 < l_1^2 - \sin^2\alpha.$$

Since $l_1 = \sin\beta$ (by definition of β, Equation 3), the right hand side can be written as $\sin^2\beta - \sin^2\alpha$, which is positive since $\alpha < \beta$ by (17). Since both sides of the inequality are positive, we can take square roots which leads to

$$|l_2 - \cos\alpha| < \sqrt{l_1^2 - \sin^2\alpha},$$

so

$$-\sqrt{l_1^2 - \sin^2\alpha} < l_2 - \cos\alpha < \sqrt{l_1^2 - \sin^2\alpha},$$

and we deduce that l_2 (which equals $l_P(\bar{\theta} - \epsilon/n)$) must satisfy

$$\cos\alpha - \sqrt{l_1^2 - \sin^2\alpha} < l_P(\bar{\theta} - \epsilon/n) < \cos\alpha + \sqrt{l_1^2 - \sin^2\alpha}. \qquad (18)$$

Comparing the left inequality of this with the definition of l_P (5), we see that the former can be written $l_P(\alpha) < l_P(\bar{\theta} - \epsilon/n)$. From (17) we note that α is less than $\bar{\theta} - \epsilon/n$ and both quantities are in the range $[0, \beta]$. The increasing property of l_P ensures that the left inequality of (18) is satisfied.

For the upper bound, note that by (15), $\bar{\theta} - \epsilon/n < \bar{\theta}$ and both quantities are in the range $[0, \beta]$. The increasing property of l_P ensures that $l_P(\bar{\theta} - \epsilon/n) < l_P(\bar{\theta})$. By the definition of l_P (5), $l_P(\bar{\theta}) < \cos\bar{\theta}$. Given $0 < \alpha < \bar{\theta} < \pi/2$ (17), the decreasing nature of cosine on this interval ensures $\cos\bar{\theta} < \cos\alpha$. Putting this together, we find $l_P(\bar{\theta} - \epsilon/n) < \cos\alpha$, so the upper bound of Inequality 18 is satisfied.

□

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