Extended Abstract
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Multifold tiles of polyominoes and convex lattice polygons

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Abstract

A family of 2-dimensional shapes $\mathcal{T}$ is called a *tiling* if they (rotating and reflecting are allowed) cover the whole plane without gaps or overlaps, and if all shapes belonging to $\mathcal{T}$ are congruent each other, then the shape is called a *tile*. We study k*-fold tilings* which are extended to cover the plane with the thickness of k folds and k*-fold tiles* which belong to it. Intuitively it means that a family of 2-dimensional shapes covers the plane such that they overlap k times at any point in the plane. Since clearly a (1-fold) tile is a k-fold tile for any positive integer k, the subjects of our research are 2-dimensional shapes with property "not a tile, but a $k(\geq 2)$-fold tile." We call a plane shape satisfying this property a *nontrivial* k*-fold tile*. In this talk, we clarify some facts as follows: first, we show that for any integer $k \geq 2$, there exists a polyomino satisfying a property that "not a h-fold tile for any positive integer $h < k$, but a k-fold tile." We also find for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones which are nontrivial k-fold tiles. Next, we prove that for any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon with an area of k that is a nontrivial k-fold tile.

Keywords: multiple tilings, k-fold tiles, polyominoes, convex polygons, thickness.

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1 Introduction

A family of 2-dimensional shapes $\mathcal{T}$ is called a *tiling* if they (rotating and reflecting are allowed) cover the whole plane without gaps or overlaps. It is said that a tiling $\mathcal{T}$ is *monohedral* [3] if any two shapes belonging to $\mathcal{T}$ are congruent. If a tiling $\mathcal{T}$ is monohedral, the unique shape in the tiling is called a *tile*. In this article, when we simply say a "tiling," it means a monohedral tiling. We study (monohedral) k*-fold tilings* which are extended to cover the plane with the thickness of k folds and k*-fold tiles* which belong to it. More specifically, a family of 2-dimensional shapes $\mathcal{T}$ is called a k-fold tiling if any point in $\mathbb{R}^2$ which is not on the boundary of any shape in $\mathcal{T}$ is covered with exactly k shapes in $\mathcal{T}$, and the shape belonging to $\mathcal{T}$ is called a k-fold tile. Since we can obtain a k-fold tiling by piling up k sheets of a tiling, it is trivial to consider a k-fold tiling with (1-fold) tiles. Hence we are interested in 2-dimensional shapes with the property: not a tile, but a $k(\geq 2)$-fold tile. We call a 2-dimensional shape having this property a *nontrivial* k*-fold tile*.

If all shapes in k-fold tiling $\mathcal{T} = \{T_1, T_2, \ldots\}$ are translates of T_1, then $\mathcal{T}$ is called a k*-fold translative tiling* [5, 6], and the shape belonging to $\mathcal{T}$ is called a k*-fold translative tile* [5, 6]. The origin of the study of multiple tilings is the one by Furtwängler [2] in 1936. He considered trivial multiple tilings as a generalization of Minkowski's conjecture [4]. As far as we know, nontrivial multiple tilings were first investigated by Bolle [1] who gave a necessary and sufficient conditions for a convex polygon to construct a multiple translative tiling with regularity (called a *multiple lattice tiling* [5, 6]). Recently, Yang and Zong [5] gave a characterization of all convex k-fold translative tiles for any $k = 2, 3, 4, 5$. Although there is various research on multiple translative tilings other than those mentioned above, there seems to be no existing research on multiple tilings allowing rotations and reflections. Therefore, the subjects of our research is such nontrivial multiple tilings. In this article, we mainly consider polyominoes: plane shapes formed by joining one or more congruent squares edge to edge, and convex lattice polygons as basic 2-dimensional shapes and present some properties.

2 Preliminaries

We introduce the following terms.

Definition 2.1. If a shape P is a k-fold tile, then k is a *tile-fold number* of P. The set of tile-fold numbers of P is denoted by $\mathrm{TFN}(P)$. If an integer k satisfies that $k \in \mathrm{TFN}(P)$ and $h \notin \mathrm{TFN}(P)$ for every positive integer $h < k$, then we call k the *minimum tile-fold number* of P, and it is denoted by $\tau^{\bullet}(P)$ [6].

Definition 2.2. Let k and h be positive integers. If an h-omino P is a nontrivial k-fold tile and there is no h'-omino that is a nontrivial k-fold tile for any positive integer $h' < h$, then h is called the *minimum size of nontrivial k-fold-tile polyomino* and P is called a *minimum-sized nontrivial k-fold-tile polyomino.*

3 Main Results

We show the following theorems.

Theorem 3.1. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^{\bullet}(P) = k$.*

Theorem 3.2. *For any integer $k \geq 2$, the minimum size of nontrivial k-fold-tile polyomino is 7, and the heptominoes C7, F7, and X7 listed in Fig. 1 are all minimum-sized nontrivial k-fold-tile polyominoes. Furthermore, the heptomino G7 in Fig. 1 is also a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$.*

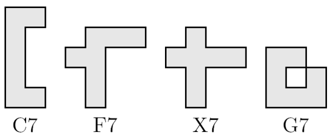

Figure 1: The heptominoes in Theorem 3.2

Theorem 3.3. *For any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon that satisfies*

(i) *a nontrivial k-fold tile and*

(ii) *the area is k.*

(iii) *it is a hexagon if $k = 5$ or 8; an octagon otherwise.*

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