# The Price of Anarchy in Network Creation Games

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## 1 *Introduction*

*Network design* is a fundamental family of problems at the intersection between computer science and operations research, amplified in importance by the sustained growth of computer networks such as the Internet. Traditionally, the goal is to find a minimum-cost (sub) network that satisfies some specified property such as $k$-connectivity or connectivity on terminals (as in the classic Steiner tree problem). Such a formulation captures the (possibly incremental) creation cost of the network, but does not incorporate the cost of actually using the network. By contrast, *network routing* has the goal of optimizing the usage cost of the network, but assumes that the network has already been created.

The *network creation game* attempts to unify the network design and network routing problems by modeling both creation and usage costs. Specifically, each node in the system can create a link (edge) to any other node, at a cost of $\alpha$. In addition to these creation costs, each node incurs a usage cost related to the distances to the other nodes. In the model introduced by Fabrikant, Luthra, Maneva, Papadimitriou, and Shenker [2003] at PODC 2003, the usage cost incurred by a node is the sum of distances to all other nodes. Equivalently, if we divide the cost

A preliminary version of this paper appeared in the *Proceedings of the 26th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing* (PODC 2007), August 2007, pages 292–298. This work was done while M. Hajiaghayi was at MIT and CMU and M. Zadimoghaddam was at Sharif University of Technology. H. Mahini was supported in part by Institute for Theoretical Physics and Mathematics (IPM) under grant number CS1384-2-01.
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(and thus $\alpha$) by the number $n$ of nodes, the usage cost is the average distance to other nodes. In another natural model, the usage cost incurred by a node is the maximum distance to all other nodes: this model captures the worst-case instead of average-case behavior of routing.

To model the dominant behavior of large-scale networking scenarios such as the Internet, we consider each node to be an agent (player) [Jackson 2003] that selfishly tries to minimize its own creation and usage cost [Fabrikant et al. 2003; Albers et al. 2006; Corbo and Parkes 2005]. In this context, the *price of anarchy* [Koutsoupias and Papadimitriou 1999; Papadimitriou 2001; Roughgarden 2002b] is the worst possible ratio of the total cost found by some independent selfish behavior and the optimal total cost possible by a centralized, social welfare maximizing solution. The price of anarchy is a well-studied concept in algorithmic game theory for problems such as load balancing, routing, and network design; see, e.g., Papadimitriou [2001; Czumaj and Vöcking [2002; Roughgarden [2002a; Fabrikant et al. [2003; Anshelevich et al. [2003; Anshelevich et al. [2004; Chun et al. [2004; Corbo and Parkes [2005; Albers et al. [2006].

*Previous work.* Three papers consider the price of anarchy in the sum network creation game. Fabrikant et al. [2003] prove an upper bound of $O(\sqrt{\alpha})$ on the price of anarchy for all $\alpha$. Lin [2003] prove that the price of anarchy is constant for two ranges of $\alpha$: $\alpha = O(\sqrt{n})$ and $\alpha \geq c\, n^{3/2}$ for some $c > 0$. Independently, Albers et al. [2006] prove that the price of anarchy is constant for $\alpha = O(\sqrt{n})$, as well as for the larger range $\alpha \geq 12\, n \lceil \lg n \rceil$.[3] In addition, Albers et al. prove a general upper bound of $15 \left(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3}\right)$. The latter bound shows the first sublinear worst-case bound, $O(n^{1/3})$, for all $\alpha$. But no better bound is known for $\alpha$ between $\omega(\sqrt{n})$ and $o(n \lg n)$. Yet $\alpha \approx n$ is perhaps the most interesting range, for it corresponds to considering the average distance (instead of the sum of distances) to other nodes to be roughly on par with link creation (effectively dividing $\alpha$ by $n$).

The *bilateral* variation on the network creation game, considered by Corbo and Parkes [2005] in PODC 2005, requires both nodes to agree before they can create a link between them. In the sum bilateral network creation game, Corbo and Parkes prove that the price of anarchy is between $\Omega(\lg \alpha)$ and $O(\sqrt{\alpha})$ for $\alpha \leq n$. Although not claimed in their paper, the proof of their upper bound also establishes an upper bound of $O(n/\sqrt{\alpha})$ for $\alpha \geq n$.

*Our results.* In the sum (unilateral) network creation game of Fabrikant et al. [2003], we prove the first $o(n^\varepsilon)$ upper bound for general $\alpha$, namely $2^{O(\sqrt{\lg n})}$. We also prove a constant upper bound for $\alpha = O(n^{1-\varepsilon})$ for any fixed $\varepsilon > 0$, substantially reducing the range of $\alpha$ for which constant bounds have not been obtained. Along the way, we also improve the constant upper bound by Albers et al. [2006] (with the lead constant of 15) to 6 for $\alpha < (n/2)^{1/2}$ and to 4 for $\alpha < (n/2)^{1/3}$.

In the max (unilateral) version, we prove that the price of anarchy is at most 2 for $\alpha \geq n$, $O(\min\{4^{\sqrt{\lg n}}, (n/\alpha)^{1/3}\})$ for $2\sqrt{\lg n} \leq \alpha \leq n$, and $O(n^{2/\alpha})$ for $\alpha < 2\sqrt{\lg n}$.

Our primary proof technique can be most closely viewed as a kind of "region

[3] As usual, $\lg n$ denotes $\log_2 n$.

<table>
<tr><th>$\alpha =$</th><th>$0$ – $1$</th><th>$1$ – $2$</th><th>$2$ – $2\sqrt{\lg n}$</th><th>$2\sqrt{\lg n}$ – $\sqrt[3]{n/2}$</th><th>$\sqrt[3]{n/2}$ – $\sqrt{n/2}$</th><th>$\sqrt{n/2}$ – $O(n^{1-\varepsilon})$</th><th>$O(n^{1-\varepsilon})$ – $n$</th><th>$n$ – $12\,n\lceil\lg n\rceil$</th><th>$12\,n\lceil\lg n\rceil$ – $n^2$</th></tr>
<tr><td>Sum unilateral</td><td>1</td><td>$\frac{4}{3}$ *</td><td colspan="2">$\le 4$ (§4)</td><td>$\le 6$ (§4)</td><td>$\Theta(1)$ (§5)</td><td colspan="2">$2^{O(\sqrt{\lg n})}$ (§6)</td><td>$\Theta(1)$ †</td></tr>
<tr><td>Max unilateral</td><td colspan="3">$O(n^{2/\alpha})$ (§7)</td><td colspan="4">$O(\min\{4^{\sqrt{\lg n}}, (n/\alpha)^{1/3}\})$ (§7)</td><td colspan="2">$\le 2$ (§7)</td></tr>
<tr><td>Sum bilateral</td><td>1</td><td colspan="6">$\Theta(\sqrt{\alpha})$ ($O$ in ‡, $\Omega$ in §8)</td><td colspan="2">$\Theta(\frac{n}{\sqrt{\alpha}})$ ($O$ in ¶, $\Omega$ in §8)</td></tr>
<tr><td>Max bilateral</td><td colspan="7">$\Theta(\frac{n}{\alpha+1})$ (§9)</td><td colspan="2">$\le 2$ (§9)</td></tr>
</table>

*[Fabrikant et al. 2003]
†[Albers et al. 2006]
‡[Corbo and Parkes 2005]
¶[Corbo and Parkes 2005, Proposition 4] claims $O(\sqrt{\alpha})$ for $\alpha < n^2$ but in fact proves $O(\min\{\sqrt{\alpha}, n/\sqrt{\alpha}\})$

Table I. Summary of best known results for all three models of network creation games and for all ranges of $\alpha$.

growing" from approximation algorithms; see, e.g., Leighton and Rao [1999].

In the (sum) bilateral network creation game of Corbo and Parkes [2005], we show that their $O(\min\{\sqrt{\alpha}, n/\sqrt{\alpha}\})$ upper bound is tight by proving a matching lower bound of $\Omega(\min\{\sqrt{\alpha}, n/\sqrt{\alpha}\})$. In the max (bilateral) version, we prove matching upper and lower bounds of $\Theta(\frac{n}{\alpha+1})$ for $\alpha \le n$, and an upper bound of 2 for $\alpha > n$.

Table I summarizes the best bounds known for each of the three models and for all ranges of $\alpha$. Interestingly, these results reveal that the price of anarchy can depend significantly on $\alpha$.

## 2 *Models*

Formally, we define four games depending on the objective (sum or max) and the consent (unilateral or bilateral). In all versions, we have $n$ players; call them $1, 2, \ldots, n$. The strategy of player $i$ is specified by a subset $s_i$ of $\{1, 2, \ldots, n\} - \{i\}$, which corresponds to the set of neighbors to which player $i$ forms a link. Together, let $s = \langle s_1, s_2, \ldots, s_n \rangle$ denote the strategies of all players.

To define the cost of a strategy, we introduce an undirected graph $G_s$ with vertex set $\{1, 2, \ldots, n\}$. In the unilateral game, $G_s$ has an edge $\{i, j\}$ if either $i \in s_j$ or $j \in s_i$. In the bilateral game, $G_s$ has an edge $\{i, j\}$ if both $i \in s_j$ and $j \in s_i$. Define $d_s(i, j)$ to be the distance (the number of edges in a shortest path) between vertices $i$ and $j$ in graph $G_s$. In the sum game, the cost incurred by player $i$ is

$$c_i(s) = \alpha\,|s_i| + \sum_{j=1}^{n} d_s(i, j),$$

and in the max game, the cost incurred by player $i$ is

$$c_i(s) = \alpha\,|s_i| + \max_{j=1}^{n} d_s(i, j).$$

In both cases, the total cost incurred by strategy $s$ is $c(s) = \sum_{i=1}^{n} c_i(s)$.

In the unilateral game, a (pure) *Nash equilibrium* is a strategy $s$ such that $c_i(s) \le c_i(s')$ for all strategies $s'$ that differ from $s$ in only one player $i$. The *price of anarchy* is then the maximum cost of a Nash equilibrium divided by the minimum cost of any strategy (called the *social optimum*).

In the bilateral game, Nash equilibria are not so interesting because the game requires coalition between two players to create an edge (in general). For example, if every player $i$ chooses the empty strategy $s_i = \emptyset$, then we obtain a Nash equilibrium inducing an empty graph $G_s$, which has an infinite cost $c(s)$. To address this issue, Corbo and Parkes [2005] use the notion of *pairwise stability* [Jackson and Wolinsky 1996]: a strategy is pairwise stable if (1) for any edge $\{i, j\}$ of $G_s$, both $c_i(s) \le c_i(s')$ and $c_j(s) \le c_j(s')$ where $s'$ differs from $s$ only in deleting edge $\{i, j\}$ from $G_s$; and (2) for any nonedge $\{i, j\}$ of $G_s$, either $c_i(s) < c_i(s')$ or $c_j(s) < c_j(s')$ where $s'$ differs from $s$ only in adding edge $\{i, j\}$ to $G_{s'}$.[4] The *price of anarchy* is then the maximum cost of a pairwise-stable strategy divided by the social optimum (the minimum cost of any strategy).

We spend the bulk of this paper (Sections 3–6) on the original version of the network creation game, sum unilateral. Then we consider the max unilateral game in Section 7, the sum bilateral game in Section 8, and the max bilateral game in Section 9. Thus we default to the unilateral game, often omitting the term, and within the unilateral game, we default to the sum version.

## 3 *Preliminaries*

In this section, we define some helpful notation and prove some basic results. Call a graph $G_s$ corresponding to a Nash equilibrium $s$ an *equilibrium graph*. In such a graph, let $d_s(u, v)$ be the length of the shortest path from $u$ to $v$. Let $N_k(u)$ denote the set of vertices with distance at most $k$ from vertex $u$, and let $N_k = \min_{v \in G} |N_k(v)|$.

We start with a lemma proved in Albers et al. [2006] about the sum unilateral game:

LEMMA 1. [Albers et al. 2006, proof of Theorem 3.2] *For any Nash equilibrium $s$ and any vertex $v_0$ in $G_s$, the cost $c(s)$ is at most $2\alpha(n-1) + n\,\mathrm{Dist}(v_0) + (n-1)^2$ where $\mathrm{Dist}(v_0) = \sum_{v \in V(G_s)} d_s(v_0, v)$.*

Now we use this lemma to relate the price of anarchy to depth in a breadth-first search (BFS) tree, a connection also used in Albers et al. [2006]. We assume henceforth that $\alpha \ge 2$, because otherwise the price of anarchy for the sum unilateral game is already known from Fabrikant et al. [2003]. In this case, it is known that the social optimum is attained by a star graph [Fabrikant et al. 2003].

LEMMA 2. *If the BFS tree in a equilibrium graph $G_s$ rooted at vertex $u$ has depth $d$, then price of anarchy is at most $d + 1$.*

**Proof:** By Lemma 1 and because the cost of the social optimum (a star) is at least $\alpha(n-1)+n(n-1)$, the price of anarchy is at most $\frac{2\alpha(n-1)+n\,\mathrm{Dist}(u)+(n-1)^2}{\alpha(n-1)+n(n-1)}$. Because the height of the BFS tree rooted at $u$ is $d$, we have $\mathrm{Dist}(u) \le (n-1)d$. Hence the price of anarchy is at most $\frac{2\alpha(n-1)+n(n-1)d+(n-1)^2}{\alpha(n-1)+n(n-1)} < \frac{2\alpha(n-1)+n(n-1)(d+1)}{\alpha(n-1)+n(n-1)} \le \max\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1)}{n(n-1)}\} = \max\{2, d+1\} = d + 1$. □

[4]To correctly handle the case of disconnected graphs, we must define $c(s)$ to consist of two parts, a finite part and an infinite part, and observe that adding an edge to a disconnected graph reduces the infinite part.

Lemma 3. *For any equilibrium graph $G_s$, $|N_2(u)| > n/(2\alpha)$ for every vertex $u$ and $\alpha \geq 1$.*

**Proof:** We can assume that the number vertices with distance more than 2 from $u$ is at least $n/2$; otherwise $|N_2(u)| > n/2 \geq n/(2\alpha)$. Let $S$ be the vertices whose distance is exactly 2 from $u$. For each vertex $v$ with $d_s(u,v) \geq 2$, we pick any one of its shortest paths to $u$ and assign $v$ to the only vertex in this path that is in $S$. The number of vertices assigned to a vertex $w \in S$ can be no more than $\alpha$, because otherwise vertex $u$ could buy edge $\{u,w\}$ and decrease its distance to each vertex assigned to $w$ by at least 1. There is one assignment for each vertex with distance at least 2 from $u$, so the total number of assignments to vertices in $S$ is more than $n/2$. Therefore, $|S| > (n/2)/\alpha = n/(2\alpha)$. □

## 4 *Improved Constant Upper Bounds for $\alpha = O(\sqrt{n})$*

In this section, we use the basic results from the previous section to improve the $O(1)$ bounds for the sum unilateral game with $\alpha = O(\sqrt{n})$ obtained by Albers et al. [2006]. Recall that their bound has a lead constant of (and is thus at least) 15. We prove an upper bound of 6 for $\alpha < (n/2)^{1/2}$ and 4 for $\alpha < (n/2)^{1/3}$.

Theorem 4. *For $\alpha < (n/2)^{1/2}$, the price of anarchy is at most 6.*

**Proof:** We prove that the height of the BFS tree rooted at an arbitrary vertex $u$ in an equilibrium graph is at most 5. Suppose for contradiction that there is a vertex $v$ at distance at least 6 from $u$. Vertex $v$ can buy $\{u,v\}$ to decrease its distance from all vertices in $N_2(u)$ by at least 1. Because vertex $v$ has not bought the edge $\{u,v\}$, we conclude that $|N_2(u)| < \alpha$. By Lemma 3, $|N_2(u)| > n/(2\alpha)$. Hence $\alpha > n/(2\alpha)$, which contradicts the hypothesis of the lemma. Therefore the height of the BFS tree rooted at $u$ is at most 5. By Lemma 2, the price of anarchy is at most $5+1=6$. □

Theorem 5. *For $\alpha < (n/2)^{1/3}$, the price of anarchy is at most 4.*

**Proof:** We prove that there is a choice of root vertex $v$ in an equilibrium graph such that the height of the BFS tree is at most 3. Let $\Delta$ be the maximum degree of a vertex in the graph and let $u$ be an arbitrary vertex. Certainly $|N_2(u)| \leq 1+\Delta+\Delta(\Delta-1) = 1+\Delta^2$. On the other hand, by Lemma 3, $|N_2(u)| > n/(2\alpha)$. By the hypothesis of the lemma, $1+\Delta^2 > n/(2\alpha) > \alpha^2$. Hence $\Delta > \alpha - 1$. Let $v$ be a vertex with degree equal to $\Delta$. Suppose for contradiction that there is vertex $u$ at distance at least 4 from $v$. Vertex $u$ can buy edge $\{u,v\}$ to decrease its distance to all vertices in $N_1(v)$ by at least 1 and thus decrease its total cost by at least $|N_1(v)| = \Delta + 1 > \alpha$, contradicting equilibrium. Therefore the height of the BFS tree rooted at $v$ is at most 3. By Lemma 2, the price of anarchy is at most $3+1=4$. □

## 5 *Constant Upper Bound for $\alpha = O(n^{1-\varepsilon})$*

In this section, we extend the range for which we know a constant upper bound on the price of anarchy in the sum unilateral game. Specifically, for $1 \leq \alpha < n^{1-\varepsilon}$ and

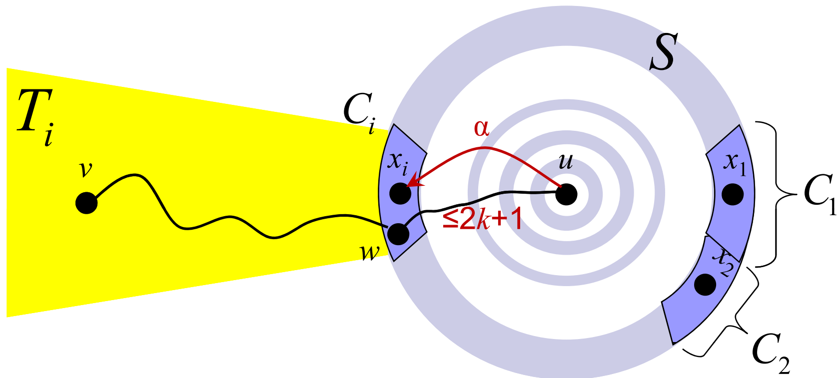


Fig. 1. Proof of Lemma 9.

$\varepsilon \geq 1/\lg n$, we prove an upper bound of $2^{O(1/\varepsilon)}$. We start with some lemmas that will be useful in this section and the next.

Lemma 6. *For any vertex $u$ in an equilibrium graph $G_s$, if $|N_k(u)| > n/2$, then $|N_{2k+2\alpha/n}(u)| \geq n$.*

**Proof:** We prove the contrapositive. Suppose $|N_{2k+2\alpha/n}(u)| < n$. Then there is a vertex $v$ with $d_s(u,v) \geq 2k+1+2\alpha/n$. For every vertex $x \in N_k(u)$, $d_s(u,x) \leq k$. By the triangle inequality $d_s(u,x)+d_s(x,v) \geq d_s(u,v)$, we have $d_s(x,v) \geq k+1+2\alpha/n$. If vertex $v$ bought the edge $\{v,u\}$, then the distance between $v$ and $x$ would decrease by at least $2\alpha/n$, so $\mathrm{Dist}(v)$ would decrease by at least $N_k(u) \cdot 2\alpha/n$. Because $v$ has not bought the edge $\{v,u\}$, we have $\alpha \geq |N_k(u)| \cdot 2\alpha/n$, i.e., $|N_k(u)| \leq n/2$. □

By setting $\alpha = n/2$ and $\alpha = 12\,n \lg n$, we conclude the following corollaries, which we will use in Theorems 10 and 12.

Corollary 7. *For any vertex $u$ in an equilibrium graph $G_s$ with $\alpha < n/2$, if $|N_k(u)| > n/2$, then $|N_{2k+1}(u)| = n$.*

Corollary 8. *For any vertex $u$ in an equilibrium graph $G_s$ with $\alpha < 12\,n \lg n$, if $|N_k(u)| > n/2$, then $|N_{2k+24\lg n}(u)| = n$.*

Lemma 9. *If $|N_k(u)| > Y$ for every vertex $u$ in an equilibrium graph $G_s$, then either $|N_{2k+3}(u)| > n/2$ for some vertex $u$ or $|N_{3k+3}(u)| > Yn/\alpha$ for every vertex $u$.*

**Proof:** If there is a vertex $u$ with $|N_{2k+3}(u)| > n/2$, then the claim is obvious. Otherwise, for every vertex $u$, $|N_{2k+3}(u)| \leq n/2$. Let $u$ be an arbitrary vertex; refer to Figure 1. Let $S$ be the set of vertices whose distance from $u$ is exactly $2k+3$. We select a subset of $S$, called *center points*, by the following greedy algorithm. First we unmark all vertices in $S$. Then we repeatedly select an unmarked vertex $x \in S$ as a center point, mark all unmarked vertices in $S$ whose distance from $x$ in $G_s$ is at most $2k$, and assign these vertices to $x$.

Suppose that we select $l$ vertices $x_1, x_2, \ldots, x_l$ as center points. We prove that $l \geq n/\alpha$. Let $C_i$ be the vertices in $S$ assigned to $x_i$. By construction, $S = \bigcup_{i=1}^{l} C_i$. We also assign each vertex $v$ at distance at least $2k+3$ from $u$ to one of these center points, as follows. Pick any one shortest path from $v$ to $u$, which contains exactly one vertex $w \in S$, and assign $v$ to the same center point as $w$. Let $T_i$ be the set of vertices assigned to $x_i$ and whose distance from $u$ is more than $2k+3$. By construction, $\bigcup_{i=1}^{l} T_i$ is the set of vertices at distance more than $2k+3$ from $u$. A shortest path from $v \in T_i$ to $u$ uses some vertex $w \in C_i$. If $u$ bought the edge $\{u, x_i\}$, then the distance between $u$ and $w$ would become at most $2k+1$. Because $d_s(u, w) = 2k+3$ in the current graph, buying edge $\{u, x_i\}$ would decrease $u$'s distance to $v$ by at least $2k+3-(1+2k) = 2$. Because $u$ has not bought the edge $\{u, x_i\}$, we conclude that $\alpha \geq 2\,|T_i|$. On the other hand, $|N_{2k+3}(u)| \leq n/2$, so $\sum_{i=1}^{l} |T_i| \geq n/2$. Therefore, $l\,\alpha \geq 2\sum_{i=1}^{l} |T_i| \geq n$ and hence $l \geq n/\alpha$.

According to the greedy algorithm, the distance between any pair of center points is more than $2k$; hence, $N_k(x_i) \cap N_k(x_j) = \emptyset$ for $i \neq j$. By the hypothesis of the lemma, $|N_k(x_i)| > Y$ for every vertex $x_i$; hence $|\bigcup_{i=1}^{l} N_k(x_i)| = \sum_{i=1}^{l} |N_k(x_i)| > l\,Y$. For every $1 \leq i \leq l$, we have $d_s(u, x_i) = 2k+3$, so vertex $u$ has a path of length at most $3k+3$ to every vertex whose distance to $x_i$ is at most $k$. Therefore, $|N_{3k+3}(u)| \geq |\bigcup_{i=1}^{l} N_k(x_i)| > l\,Y \geq Yn/\alpha$. □

THEOREM 10. *For $\varepsilon \geq 1/\lg n$ and $1 \leq \alpha < n^{1-\varepsilon}$, the price of anarchy is at most $4.667 \cdot 3^{\lceil 1/\varepsilon \rceil} + 8$.*

**Proof:** Consider an equilibrium graph $G_s$. Let $X = n/\alpha > n^{\varepsilon}$. Define $a_1 = 2$ and $a_i = 3a_{i-1} + 3$ for all $i > 1$. By Lemma 3, $N_2(u) > n/(2\alpha) = X/2$ for all $u$. By Lemma 9, for each $i \geq 1$, either $N_{2a_i+3}(v) > n/2$ for some vertex $v$ or $N_{a_{i+1}} \geq (n/\alpha)\,N_{a_i} = X\,N_{a_i}$. Let $j$ be the least number for which $|N_{2a_j+3}(v)| > n/2$ for some vertex $v$. By this definition, for each $i < j$, $N_{a_{i+1}} > (n/\alpha)\,N_{a_i} = X\,N_{a_i}$. Because $N_{a_1} = N_2 > X/2$, we obtain that $N_{a_i} > X^i/2$ for every $i \leq j$. On the other hand, $X^j/2 < N_{a_j} \leq n$, so $X^j < 2n$. Therefore $j < \frac{1}{\varepsilon}(1 + 1/\lg n)$. Because $\varepsilon \leq 1/\lg n$, we must have $j < \frac{1}{\varepsilon} + 1$. Because $j$ is an integer, $j \leq \lceil 1/\varepsilon \rceil$. So $|N_{2a_{\lceil 1/\varepsilon \rceil}+3}(v)| \geq |N_{2a_j+3}(v)| > n/2$. Because $\alpha < n^{1-\varepsilon}$ and $\varepsilon \geq 1/\lg n$, we have $\alpha < n/2$, and thus we can use Corollary 7. By Corollary 7, $|N_{4a_{\lceil 1/\varepsilon \rceil}+7}(v)| = n$. Hence, the height of the BFS tree rooted at vertex $v$ is at most $4a_{\lceil 1/\varepsilon \rceil} + 7$. By Lemma 2, the price of anarchy is at most $4a_{\lceil 1/\varepsilon \rceil} + 8$. Solving the recurrence relation $a_i = 3a_{i-1} + 3$ with $a_1 = 2$, we obtain that $a_i = \frac{7}{6}3^i - \frac{3}{2} < \frac{7}{6}3^i$. Therefore the price of anarchy is at most $4\,\frac{7}{6}\,3^{\lceil 1/\varepsilon \rceil} + 8 \leq 4.667 \cdot 3^{\lceil 1/\varepsilon \rceil} + 8$. □

## 6 $o(n^{\varepsilon})$ *Upper Bound for* $\alpha < 12\,n \lg n$

In this section, we prove the first $o(n^{\varepsilon})$ bound for the sum unilateral game with $\alpha$ between $\Omega(n)$ and $o(n \lg n)$. Specifically, we show an upper bound of $2^{O(\sqrt{\lg n})}$. First we need the following lemma.

LEMMA 11. *If $|N_k(u)| > Y$ for every vertex $u$ in an equilibrium graph $G_s$, then either $|N_{4k+1}(u)| > n/2$ for some vertex $u$ or $|N_{5k+1}(u)| > Y k\,n/\alpha$ for every vertex $u$.*

**Proof:** The proof is similar to the proof of Lemma 9. If there is a vertex $u$ with $|N_{4k+1}(u)| > n/2$, then the claim is obvious. Otherwise, for every vertex $u$, $|N_{4k+1}(u)| \leq n/2$. Let $u$ be an arbitrary vertex. Let $S$ be the set of vertices whose distance from $u$ is $4k+1$. We select a subset of $S$, called *center points*, by the following greedy algorithm. First we unmark all vertices in $S$. Then we repeatedly select an unmarked vertex $x \in S$ as a center point, mark all unmarked vertices in $S$ whose distance from $x$ is at most $2k$, and assign these vertices to $x$.

Suppose that we select $l$ vertices $x_1, x_2, \ldots, x_l$ as center points. We prove that $l \geq k\, n/\alpha$. Let $C_i$ be the vertices in $S$ assigned to $x_i$. By construction, $S = \bigcup_{i=1}^{l} C_i$. We also assign each vertex $v$ with distance at least $4k+1$ from $u$ to one of these center points. Pick any one shortest path from $v$ to $u$, which contains exactly one vertex $w \in S$, and assign $v$ to the same center point as $w$. Let $T_i$ be the set of vertices assigned to $x_i$ and whose distance from $u$ is more than $4k+1$. By construction, $\bigcup_{i=1}^{l} T_i$ is the set of vertices at distance more than $4k+1$ from $u$. The shortest path from $v \in T_i$ to $u$ uses some vertex $w \in C_i$. If $u$ bought the edge $\{u, x_i\}$, then the distance between $u$ and $w$ would become at most $2k+1$. Because $d_s(u,w) = 4k+1$, buying edge $\{u, x_i\}$ would decrease $u$'s distance to $v$ by at least $4k+1-(2k+1) = 2k$. Because $u$ has not bought the edge $\{u, x_i\}$, we conclude that $\alpha \geq 2k|T_i|$. On the other hand, $|N_{4k+1}(u)| \leq n/2$ and $\sum_{i=1}^{l} |T_i| \geq n/2$. Therefore, $l\,\alpha \geq 2k \sum_{i=1}^{l} |T_i| \geq k\,n$ and hence $l \geq k\,n/\alpha$.

According to the greedy algorithm, the distance between any pair of center points is more than $2k$; hence, $N_k(x_i) \cap N_k(x_j) = \emptyset$ for $i \neq j$. By the hypothesis of the lemma, $|N_k(x_i)| > Y$ for every vertex $x_i$; hence $|\bigcup_{i=1}^{l} N_k(x_i)| = \sum_{i=1}^{l} |N_k(x_i)| > l\,Y$. For every $i \leq l$, we have $d_s(u, x_i) = 4k+1$, so vertex $u$ has a path of length at most $5k+1$ to every vertex whose distance to $x_i$ is at most $k$. Therefore, $|N_{5k+1}(u)| \geq |\bigcup_{i=1}^{l} N_k(x_i)| > l\,Y \geq Y k\, n/\alpha$. □

THEOREM 12. *For $1 \leq \alpha < 12\, n \lg n$, the price of anarchy is $O(5^{\sqrt{\lg n}} \lg n)$.*

**Proof:** The proof is similar to the proof of Theorem 10. Let $Z = 12 \lg n$. By the hypothesis of the theorem, $\alpha/n < 12 \lg n = Z$. Because any equilibrium graph $G_s$ is connected, $|N_Z| > Z$. By Lemma 11, either $|N_{4k+1}(v)| > n/2$ for some vertex $v$ or $N_{5k+1} \geq (n/\alpha)k\, N_k$. Define $a_0, a_1, \ldots$ by the recurrence relation $a_i = 5\, a_{i-1} + 1$ with $a_0 = Z$. By induction, $a_i > Z\, 5^i$. Suppose $j$ is the least number for which $|N_{4a_j+1}(v)| > n/2$. By this definition, and because $n/\alpha > 1/Z$, we obtain that $N_{a_{i+1}} \geq (n/\alpha) a_i N_{a_i} > 5^i N_{a_i}$ for each $i < j$. From these inequalities we derive that $N_{a_j} > 5^{\sum_{i=0}^{j-1} i}$. But $N_{a_j} \leq n$, so $\sum_{i=1}^{j-1} i = j(j-1)/2 \leq \log_5 n$. This inequality implies that $j < 1 + \sqrt{2 \log_5 n} < 1 + \sqrt{\lg n}$. By Corollary 8, the height of the BFS tree rooted at $v$ is at most $2(4\, a_{1+\sqrt{\lg n}} + 1) + 24 \lg n$. Solving the recurrence relation, we obtain that $a_j = O(5^j \lg n)$. By Lemma 2, the price of anarchy is $O(5^{\sqrt{\lg n}} \lg n)$. □

## 7 *Upper Bounds for Max Unilateral Game*

In this section, we introduce and analyze the natural max variation on the (unilateral) network creation game. This problem is motivated by players forming a network with guaranteed worst-case performance subject to budget constraints. We prove that the price of anarchy is at most 2 for $\alpha \geq n$, $O(\min\{4^{\sqrt{\lg n}}, (n/\alpha)^{1/3}\})$ for $2\sqrt{\lg n} \leq \alpha \leq n$, and $O(n^{2/\alpha})$ for $\alpha < 2\sqrt{\lg n}$.

Let $D_v$ be the distance of the farthest vertex from $v$ and $N_k^{=}(v)$ be the set of vertices whose distance to $v$ is exactly $k$. Thus the cost incurred by vertex $v$ is

$\alpha\, e_v + D_v$, where $e_v$ is the number of edges that $v$ has bought.

LEMMA 13. *Any equilibrium graph $G_s$ has no cycle of length less than $\alpha + 2$.*

**Proof:** Suppose for contradiction that there is cycle $C$ with $|C| < \alpha + 2$. Let $\{u, v\}$ be an edge of this cycle. Suppose by symmetry that $u$ bought this edge. If vertex $u$ removed the edge, it would decrease its buying cost $e_u$ by $\alpha$ and increase its distance cost $D_u$ by at most $|C| - 2 < \alpha$. Hence it is cost effective for $u$ to remove this edge, contradicting equilibrium. □

THEOREM 14. *For $\alpha \geq n$, the price of anarchy is at most 2.*

**Proof:** By Lemma 13, the equilibrium graph $G_s$ has no cycle of length at most $n$, which implies that $G$ is a tree. The cost of $s$ is at most $\alpha(n-1) + \sum_{v \in V(G)} D_v \leq \alpha(n-1) + n(n-1) \leq 2\alpha(n-1)$. On the other hand, the cost of the social optimum is at least $\alpha(n-1)$. Therefore the price of anarchy is at most $\frac{2\alpha(n-1)}{\alpha(n-1)} = 2$. □

Lemma 13 gives us a lower bound on the *girth* (length of the shortest cycle) of an equilibrium graph. We use the following result of Dutton and Brigham [1991] to relate the number of edges to the girth:

LEMMA 15. [Dutton and Brigham 1991] *An $n$-vertex graph of girth at least $g$, where $g$ is odd, has $O(n^{1+2/(g-1)})$ edges.*

LEMMA 16. *The number of edges in an equilibrium graph is $O(n^{1+2/\alpha})$.*

**Proof:** By Lemma 13, the girth of the equilibrium graph $G$ is at least $\alpha + 2$ and hence at least $\lceil \alpha + 2 \rceil$. So by Lemma 15, the number of edges in $G$ is $O(n^{1+2/(\lceil \alpha+2 \rceil - 1)})$ for odd values of $\lceil \alpha \rceil$, and the number of edges is $O(n^{1+2/(\lceil \alpha+1 \rceil - 1)})$ for even values of $\lceil \alpha \rceil$. □

LEMMA 17. *For a vertex $v$ in an equilibrium graph, and for $k \leq D_v/2$, we have $|N^{=}_{k+1}(v)| \geq k/\alpha$ and $|N_{k+1}(v)| > k^2/(2\alpha)$.*

**Proof:** If vertex $v$ bought the $|N^{=}_{k+1}(v)|$ edges connecting $v$ to the vertices in $N^{=}_{k+1}(v)$, the distance from $v$ to all vertices outside $N_k(v)$ decreases by $k$, while possibly not changing the distance from $v$ to vertices in $N_k(v)$. Thus the new value of $D_v$ would become at most $\max\{k, D_v - k\} = D_v - k$ because $k \leq D_v/2$. Thus $D_v$ would decrease by at least $D_v - (D_v - k) = k$. The cost of buying these edges is $|N^{=}_{k+1}(v)|\, \alpha$. Because $v$ has not bought these edges, we conclude that this cost is at least $k$. Hence $|N^{=}_{k+1}(v)| \geq k/\alpha$. Therefore $|N_{k+1}(v)| \geq \sum_{i=1}^{k} (i/\alpha) > k^2/(2\alpha)$. □

LEMMA 18. *The diameter of an equilibrium graph $G_s$, $\mathrm{diam}(G_s)$, is $O((n\,\alpha^2)^{1/3})$.*

**Proof:** Set $k = \lfloor \mathrm{diam}(G_s)/4 \rfloor - 1$. Let $v$ be some vertex with $D_v = \mathrm{diam}(G_s)$. Similar to the proof of Lemma 11, we select a subset of vertices as *center points* by the following greedy algorithm. First we unmark all vertices in $G_s$. Then we repeatedly select an unmarked vertex $x$ as a center point, and mark all unmarked vertices whose distances are at most $2k$ from $x$.

Suppose that we select $l$ vertices $x_1, x_2, \dots, x_l$ as center points. By construction, every vertex in the graph has distance at most $2k$ to some center point. If vertex $v$ bought the $l$ edges $\{v, x_1\}, \{v, x_2\}, \dots, \{v, x_l\}$, it would decrease $D_v$ by at least $D_v - (2k+1) \geq (4k+4) - (2k+1) > 2k$. Buying these edges costs $\alpha\, l$. Because $v$ has not bought these edges, we have $\alpha\, l \geq 2k$. On the other hand, according to the greedy algorithm, the distance between any pair of center points is at least $2k+1$. Thus we have $N_k(x_i) \cap N_k(x_j) = \emptyset$ for $1 \leq i < j \leq l$. Therefore, $|\bigcup_{i=1}^{l} N_k(x_i)| = \sum_{i=1}^{l} |N_k(x_i)|$, which should be less than or equal to $n$. By Lemma 17, we have $n \geq |\bigcup_{i=1}^{l} N_k(x_i)| = \sum_{i=1}^{l} |N_k(x_i)| \geq l(k-1)^2/(2\alpha)$, which means that $l \leq 2\, n\, \alpha/(k-1)^2$. Combining with $\alpha\, l \geq 2\, k$, we obtain that $2k \leq 2\, n\, \alpha^2/(k-1)^2$, i.e., $k(k-1)^2 \leq n\, \alpha^2$. Therefore $\operatorname{diam}(G_s) = O((n\, \alpha^2)^{1/3})$. □

THEOREM 19. *For $6 \leq \alpha < n$, the price of anarchy is $O((n/\alpha)^{1/3})$.*

**Proof:** By Lemma 16, the number of edges in $G_s$ is $O(n^{1+2/\alpha})$. By Lemma 18, $\operatorname{diam}(G_s) = O((n\, \alpha^2)^{1/3})$. Hence the cost of $G_s$ is at most $\alpha\, O(n^{1+2/\alpha}) + n\, O((n\alpha^2)^{1/3})$. On the other hand, the cost of the social optimum is $\Omega(n\, \alpha)$. Therefore the price of anarchy is $O(n^{2/\alpha} + (n\, \alpha^2)^{1/3}/\alpha) = O(n^{2/\alpha} + (n/\alpha)^{1/3}) = O((n/\alpha)^{1/3})$ because $\alpha \geq 6$. □

LEMMA 20. *If $|N_k(u)| > Y$ for every vertex $u$ in an equilibrium graph $G_s$, then either $D_u \leq 5\, k$ for some vertex $u$ or $|N_{4k+1}(u)| > Y\, k/\alpha$ for every vertex $u$.*

**Proof:** The proof is similar to the proof of Lemma 11. If there is a vertex $u$ with $D_u \leq 5\, k$, then the claim is obvious. Otherwise, for every vertex $u$, we have $D_u > 5k$. Let $u$ be an arbitrary vertex. Let $S$ be the set of vertices whose distance from $u$ is $3k+1$. We select a subset of $S$, called *center points*, by the following greedy algorithm. First we unmark all vertices in $S$. Then we select an unmarked vertex $x \in S$ as a center point, mark all unmarked vertices in $S$ whose distance from $x$ is at most $2k$, and assign these vertices to $x$.

Suppose that we select $l$ vertices $x_1, x_2, \dots, x_l$ as center points. We prove that $l \geq k/\alpha$. If vertex $u$ bought the $l$ edges $\{u, x_1\}, \{u, x_2\}, \dots, \{u, x_l\}$, it would decrease $D_u$ by at least $D_u - \max\{D_u - k, 3k+1\} = D_u - (D_u - k) = k$. Because $u$ has not bought these edges, we must have $l\, \alpha \geq k$.

According to the greedy algorithm, the distance between any pair of center points is more than $2k$; hence $N_k(x_i) \cap N_k(x_j) = \emptyset$ for $i \neq j$. By the hypothesis of the lemma, $|N_k(x_i)| > Y$ for every vertex $x_i$; hence $|\bigcup_{i=1}^{l} N_k(x_i)| = \sum_{i=1}^{l} |N_k(x_i)| > l\, Y$. For every $i \leq l$, we have $d_s(u, x_i) = 3k+1$, so vertex $u$ has a path of length at most $4k+1$ to every vertex whose distance to $x_i$ is at most $k$. Therefore $|N_{4k+1}(u)| \geq |\bigcup_{i=1}^{l} N_k(x_i)| > l\, Y \geq Y\, k/\alpha$. □

THEOREM 21. *The price of anarchy is $O(4^{\sqrt{\lg n}} + n^{2/\alpha})$.*

**Proof:** The proof is similar to the proof of Theorem 12. Let $Z = \alpha$. Consider an arbitrary vertex $v$. Because any equilibrium graph is connected, $|N_Z| > Z$. By Lemma 20, either $D_v \leq 5k$ for some vertex $v$ or $N_{4k+1} > (k/\alpha)N_k$ for every vertex $v$. Define the numbers $a_0, a_1, \dots$ using the recurrence relation $a_i = 4\, a_{i-1} + 1$ with $a_0 = Z$. By induction, $a_i \geq Z\, 4^i$. Suppose that $j$ is the least number for which

$D_v \leq 5\,a_j$ for some vertex $v$. By this definition, $N_{a_{i+1}} \geq (a_i/\alpha)N_{a_i} \geq 4^i N_{a_i}$ for each $i < j$. From these inequalities we derive that $N_{a_j} > 4^{\sum_{i=0}^{j-1} i}$. But $|N_{a_j}(v)| \leq n$, so $\sum_{i=1}^{j-1} i = j(j-1)/2 \leq \log_4 n$. This inequality implies that $j \leq 1 + \sqrt{2\log_4 n} = 1 + \sqrt{\lg n}$. Also, $D_v \leq 5\,a_j$. Solving the recurrence relation, $a_j = O(4^j \alpha)$. Thus $D_v$ and therefore $\mathrm{diam}(G_s)$ are $O(4^{\sqrt{\lg n}}\alpha)$.

On the other hand, by Lemma 16, the number of edges in the graph is $O(n^{1+2/\alpha})$. Hence the cost of the graph is at most $n\,\mathrm{diam}(G_s) + \alpha\,O(n^{1+2/\alpha}) = O(n\,\alpha\,4^{\sqrt{\lg n}} + \alpha\,n^{1+2/\alpha})$. The cost of the social optimum is at least $\alpha(n-1) = \Omega(n\,\alpha)$. Therefore the price of anarchy is $O(4^{\sqrt{\lg n}} + n^{2/\alpha})$. □

We conclude the following corollaries from Theorem 21:

Corollary 22. *For $\alpha > \sqrt{\lg n}$, the price of anarchy is $O(4^{\sqrt{\lg n}})$.*

Corollary 23. *For $\alpha \leq \sqrt{\lg n}$, the price of anarchy is $O(n^{2/\alpha})$.*

## 8 *Tight Lower Bounds for Sum Bilateral Game*

In this section, we prove tight lower bounds on the sum version of the bilateral network creation game. Although not stated explicitly, [Corbo and Parkes 2005, Proposition 4] establishes an upper bound of $O(\frac{n\sqrt{\alpha}}{\alpha+n})$. For $\alpha < n$, this upper bound is $O(\sqrt{\alpha})$; for $\alpha > n$, this upper bound is $O(n/\sqrt{\alpha})$. We prove matching lower bounds using the following lemma:

Lemma 24. *There is a tree $T_{d,k}$ with $n = 1 + dk$ vertices and the total cost greater than $2\alpha(n-1) + 2\binom{k}{2}d^2(d+1) = 2\alpha(n-1) + k(k-1)d^2(d+1)$, for $\alpha > 2d^2$.*

**Proof:** We construct the tree $T_{d,k}$ as follows. Put a vertex $r$ as root in the tree. Attach $k$ paths $P_1, P_2, \ldots, P_k$ each of which is of length $d$ to $r$. So there are $k$ paths with $d$ edges which all of them have $r$ as one of their endpoints. Note that there are $kd$ edges and $n = 1 + kd$ vertices in this tree.

Because all edges are cut edges, there is no edge $\{u, v\}$ that $u$ and $v$ want to remove. On the other hand, it is sufficient to prove that no edge will be added, to conclude that $T_{d,k}$ is an equilibrium graph. For sake of contradiction, suppose there is a pair of vertices such as $a$ and $b$ such that both of them want to build the edge $\{a, b\}$ and pay $\alpha$ for the edge. Suppose that $a \in P_i$ and $b \in P_j$, note that $i$ may be equal to $j$ which means that $a$ and $b$ are from one path. Let $d_a$ and $d_b$ be the length of the paths from $r$ to $a$ and $b$ in $T_{d,k}$ respectively. One of the $d_a$ and $d_b$ might be zero which means that one of the $a$ and $b$ is the root $r$. Without loss of generality, assume that $d_a \leq d_b$. The length of the shortest path from $a$ to the vertices outside of path $P_j$ would not decrease with this edge. On the other hand its distance to vertices in $P_j$ is decreased at most $|P_i| + |P_j| = 2d$. Because there are $d$ vertices in $P_j$, vertex $a$ has decreased its sum of distances to the other $n-1$ vertices at most $d(2d) = 2d^2$. But $a$ paid $\alpha$ to buy this edge. So $\alpha \leq 2d^2$ which is a contradiction. Therefore $T_{d,k}$ is an equilibrium graph. The cost of $T_{d,k}$ is $2\alpha(n-1) + \sum_{u,v \in V(T)} d_s(u,v)$. The sum of the distances between vertices in different paths is obviously less than the sum of distances over all pairs of vertices. The distance between two vertices from two different paths is equal to the sum of their distances to $r$. So the cost of the tree is at least $2\alpha(n-1) + \sum_{i \neq j} \sum_{a=1}^{d} \sum_{b=1}^{d} a + b = 2\alpha(n-1) + k(k-1)d^2(d+1)$. □

THEOREM 25. *The price of anarchy is at least $\Omega(\sqrt{\alpha})$ when $\alpha < n$.*

**Proof:** By Lemma 24 and setting $d = \sqrt{\alpha/2} - 1$, we reach an equilibrium graph with cost at least $d^2(d+1)k(k-1) = \Omega(n^2 d)$. The social optimum (a star) has cost at most $2\alpha(n-1) + 2n^2 = O(n^2)$. Therefore the price of anarchy is at least $\Omega(d) = \Omega(\sqrt{\alpha})$. □

THEOREM 26. *The price of anarchy is at least $\Omega(n/\sqrt{\alpha})$ when $\alpha \geq n$.*

**Proof:** Again by Lemma 24 and setting $d = \sqrt{\alpha/2} - 1$, we reach an equilibrium graph with cost at least $d^2(d+1)k(k-1) = \Omega(n^2 d)$. The social optimum (a star) has cost at most $2\alpha(n-1) + 2n^2 = O(\alpha\, n)$. Therefore the price of anarchy is at least $\Omega(n\, d/\alpha) = \Omega(n/\sqrt{\alpha})$. □

## 9 *Tight Bounds for Max Bilateral Game*

In this section, we prove tight bounds on the max version of the bilateral network creation game.

THEOREM 27. *The price of anarchy is at least $\Omega(\frac{n}{\alpha+1})$ for any $\alpha$.*

**Proof:** Consider tree $T_{d,3}$ as it is defined in Lemma 24 which has $n = 1 + 3d$ vertices. Again because all edges are cut edges, there is no edge $\{u, v\}$ that $u$ and $v$ want to remove. On the other hand, suppose there is a pair of vertices such as $a$ and $b$ such that both of them want to build the edge $\{a, b\}$ and pay $\alpha$ for the edge. Assume that $a \in P_i$ and $b \in P_j$, note that $i$ may be equal to $j$ which means that $a$ and $b$ are from one path. Let $d_a$ and $d_b$ be the length of the paths from $r$ to $a$ and $b$ in $T_{d,3}$ respectively. One of the $d_a$ and $d_b$ might be zero which means that one of the $a$ and $b$ is the root $r$. Without loss of generality, assume that $d_a \leq d_b$. After adding this edge, the height of the BFS tree rooted at $a$ remains $d_a + d$ which has not changed. So vertex $a$ has no interest to buy this edge. Therefore $T_{d,3}$ is an equilibrium graph.

The cost of $T_{d,3}$ is at least $\Omega(2\alpha(n-1)+nd) = \Omega(\alpha\, n+n^2) = \Omega(n^2)$. Because the cost of the social optimum is $O(\alpha\, n+n)$, the price of anarchy is at least $\Omega(\frac{n}{\alpha+1})$. □

THEOREM 28. *The price of anarchy is $O(\frac{n}{\alpha+1})$ for any $\alpha \leq n$, and at most 2 for $\alpha > n$.*

**Proof:** Because players can force the removal of edges in the bilateral version, Lemma 13 still holds by the same proof. Thus we can apply Theorem 14 to obtain the desired upper bound of 2 in the case $\alpha > n$. When $\alpha \leq n$, we can apply Lemma 13 to derive that no cycle has length less than $\alpha + 2$ in an equilibrium graph. Because the number of edges in the graph is also at most $n(n-1)/2$, there are $O(\min\{n^2, n^{1+2/\alpha}\})$ edges in the equilibrium graph. On the other hand, the height of the BFS tree rooted at any vertex of the graph is at most $n$. So the cost is $O(\min\{n^2, n^{1+2/\alpha}\}\alpha + n^2)$. Knowing the fact that the social optimum cost is at least $\Omega(\alpha\, n+n)$, the price of anarchy is $\frac{O(n^{1+2/\alpha}\alpha+n^2)}{\Omega(n\alpha+n)} = O(n^{2/\alpha} + \frac{n}{\alpha+1}) = O(\frac{n}{\alpha+1})$ for $2 < \alpha \leq n$ and $\frac{O(n^2\alpha+n^2)}{\Omega(n\alpha+n)} = O(n) = O(\frac{n}{\alpha+1})$ for $\alpha \leq 2$. Therefore, for any $\alpha \leq n$, the price of anarchy is $O(\frac{n}{\alpha+1})$. □

## 10 *Conclusion*

In this paper, we have bounded the price of anarchy in four different network creation games. We have significantly improved the bounds for the sum unilateral game, introduced the new max game, and completely resolved the bilateral games. We conjecture that the correct bound on the price of anarchy for the sum unilateral game is $\Theta(1)$. For the max unilateral game, a general constant bound does not seem impossible; in any case, it would be interesting to determine the optimal bound.

One interesting generalization of all of these games is when only some links can possibly be created (because of physical limitations, for example). More precisely, we are given a (connected) graph of the allowable edges, and the players correspond to nodes in this graph. In this case, the socially optimal strategy is no longer simply a clique or a star, and it is not even clear whether it can be computed in polynomial time. Thus the price of stability (the *minimum* cost of a Nash equilibrium divided by the social optimum) also becomes of interest.

## ACKNOWLEDGMENT

We thank the anonymous referees for helpful comments.

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RECEIVED FEBRUARY 2009; REVIEWED AUGUST 2009; ACCEPTED SEPTEMBER 2009