Extended Abstract
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Orthogonal Fold & Cut

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Abstract

We characterize the shapes that can be produced by "Orthogonal Fold & Cut": folding a rectangular sheet of paper on vertical and horizontal creases, and then making a single straight cut. We also solve a handful of simpler related problems: Orthogonal Fold & Punch, 1-Dimensional Fold & Cut, Signed 1-Dimensional Fold & Cut, and 1-Dimensional Interval Fold & Cut.

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1 Introduction

Given a rectangular piece of paper marked with some line segments called *cuts*, *Fold & Cut* asks us to fold the paper into a flat configuration and then make a single infinite cut to cut exactly along the marked line segments. More formally, we wish to fold the paper such that the cuts all lie on the same line, and no other part of the paper is on the line. It turns out that Fold & Cut is solvable for any (finite) set of line segments [2, 3].

Orthogonal Fold & Cut asks the same question, but we are now restricted to *orthogonal* folds, meaning every crease must be parallel to an edge of the rectangle. Again we use only a single cut, which may be at any angle. It is easy to construct sets of cuts which cannot be obtained this way, but some complicated-looking shapes still can be. Figures 1 and 2 show example instances and their solutions.

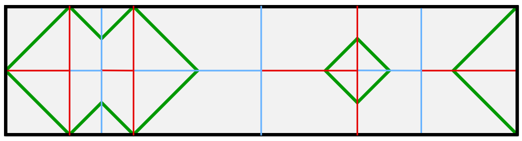

Figure 1: An example instance of Orthogonal Fold & Cut (bold green lines), and the crease pattern our algorithm generates (thin red mountains and blue valleys).

We characterize when Orthogonal Fold & Cut is solvable. In addition, we provide characterizations for several simpler related problems:

- Given a line segment of paper with marked *cut points*, *1-Dimensional Fold & Cut* asks us to fold the line segment and make a single (0-dimensional) cut that hits exactly the cut points. We do not allow folds at cut points.
- In *Signed 1-Dimensional Fold & Cut*, each cut point is given a sign, either positive or negative, and we are asked to have the paper oriented according to the sign at each cut point. Intuitively, the cut points are marked on only one side, and they need to all be face up.

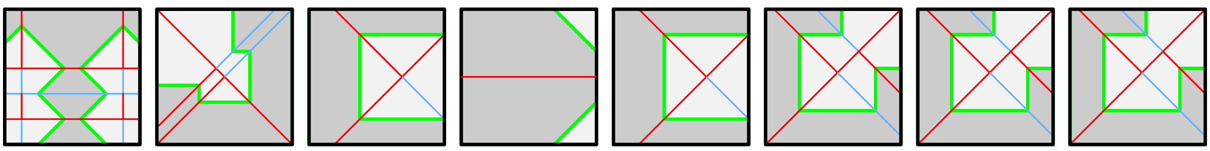

Figure 2: A few letters from our Orthogonal Fold & Cut mathematical font, in the notation of Figure 1.

- In *1-Dimensional Interval Fold & Cut*, we are asked to fold a line segment so that specified *cut intervals* lie on a common interval in the folded state, and no other part of the line segment lies on this interval. Additionally, the cut intervals contain some marked creases which are required to be folded, and no other folds can be made within any cut interval.
- In *Orthogonal Fold & Punch*, we are given a rectangular piece of paper marked with points called *holes*, and are asked to fold it flat using only orthogonal folds and punch out a single point to remove exactly the holes. The nonorthogonal Fold & Punch problem always has a solution [1].

2 Results

Our characterizations take the form of defining an easy-to-compute "canonical" crease pattern and showing that, if the problem has any solution, then the canonical one works.

Lemma 2.1. *1-Dimensional Fold & Cut is always solvable.*

Lemma 2.2. *Signed 1-Dimensional Fold & Cut is solvable if and only if the signs of cuts alternate.*

Refer to Figure 3. The canonical crease pattern for Signed and unsigned 1D Fold & Cut puts a crease point at the midpoint between each consecutive pair of cut points. For 1D Interval Fold & Cut, the canonical crease pattern has a crease at each required crease and at the midpoint between each pair of consecutive cut intervals.

Lemma 2.3. *An instance of 1-Dimensional Interval Fold & Cut is solvable if and only if the canonical crease pattern is a solution.*

Figure 3: The canonical solution for an instance of 1-Dimensional Fold & Cut, and the canonical solution for an instance of 1-Dimensional Interval Fold & Cut.

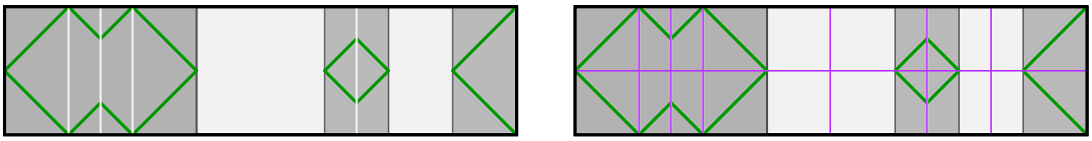

Figure 4: An instance of Orthogonal Fold & Cut with vertical bands shaded (left), and its (unsigned) canonical crease pattern (right).

For the main problem, Orthogonal Fold & Cut, we first note that the slopes of cut lines in a solvable instance must all be $\pm\alpha$ for a common α. The degenerate cases $\alpha = 0$ and ∞ are easy, and otherwise we can scale the paper so $\alpha = 1$, as in Figure 4. Next, local constraints imply that some intervals, which we call *bands*, must not have vertical creases, and that there are vertical creases at some particular positions. We call other intervals *stripes*, including zero-width stripes at positions required to have creases. Our main result characterizes the solvable instances of Orthogonal Fold & Cut in terms of stripes and bands, by factoring the instance into two othogonal instances of 1D Interval Fold & Cut.

Theorem 2.4. *Consider an instance of Orthogonal Fold & Cut on rectangular paper in which every crease has finite nonzero slope $\pm\alpha$. We call the crease pattern which puts one vertical (resp. horizontal) crease at the center of each vertical (resp. horizontal) stripe, including zero-width stripes, the* canonical *crease pattern. If the instance is solvable, then the canonical crease pattern is a solution.*

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