Grid Vertex-Unfolding Orthostacks

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Abstract. An algorithm was presented in [BDD+98] for unfolding orthostacks into one piece without overlap by using arbitrary cuts along the surface. It was conjectured that orthostacks could be unfolded using cuts that lie in a plane orthogonal to a coordinate axis and containing a vertex of the orthostack. We prove the existence of a vertex-unfolding using only such cuts.

1 Introduction

A long-standing open question is whether every convex polyhedron can be *edge unfolded*—cut along some of its edges and unfolded into a single planar piece without overlap [She75,O'R98,Dem00,DO05]. A related open question asks whether every polyhedron without boundary[4] (not necessarily convex but forming a closed surface) can be *generally unfolded*—cut along its surface (not just along edges) and unfolded into a single planar piece without overlap. Biedl et al. [BDD+98] made partial progress on both of these problems in the context of *orthostacks*. An orthostack is an orthogonal polyhedron for which every horizontal planar slice is connected, and for which the interior of the polyhedron is a connected solid. Thus, every horizontal planar slice of an orthostacks's interior is a simple polygon. Biedl et al. showed that not all orthostacks can be edge unfolded (see Figure 1), but that all orthostacks can be generally unfolded. In their general unfoldings, all cuts are parallel to coordinate axes, but many of the cuts do not lie in coordinate planes that contain polyhedron vertices. Given the lack of pure edge unfoldings, the closest analog we can hope for with (nonconvex)

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[4] For the purposes of this problem, a *polyhedron without boundary* is an abstract polyhedral complex without boundary, i.e., a set of polygons and a definition of incidence between polygons such that every edge is incident to exactly two polygons and every two polygons meet at either a common vertex, a common edge, or not at all. Note that a polyhedron is treated as a surface throughout this paper.

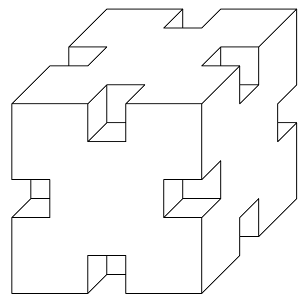

Fig. 1. This orthostack is not edge-unfoldable [BDD+98].

orthostacks is to find *grid unfoldings* in which every cut is in a coordinate plane that contains a polyhedron vertex. In other words, a grid unfolding is an edge unfolding of the refined ("gridded") polyhedron in which we slice along every coordinate plane containing a polyhedron vertex. Biedl et al. [BDD+98] asked whether all orthostacks can be grid unfolded.

We make partial progress on this problem by showing that every orthostack can be *grid vertex-unfolded*, i.e., cut along some of the grid lines and unfolded into a vertex-connected planar piece without overlap. Vertex-unfoldings were introduced in [DEE+02,DEE+03]; the difference from edge unfoldings is that faces can remain connected along single points (vertices) instead of having to be connected along whole edges.

2 A Grid Vertex Unfolding

Given an orthostack K, let $z_0 < z_1 < \cdots < z_n$ be the distinct z coordinates of vertices of K. Subdivide the faces of K by cutting along every plane perpendicular to a coordinate axis that passes through a vertex of K. This subdivision *rectangulates* K. We use the term *rectangle* to refer to one element of this facial subdivision, while *face* refers to a maximal connected set of coplanar rectangles. We use *up* and *down* to refer to the z dimension, and use *left* and *right* to refer to the x dimension.

2.1 Rectangle Categorization

We partition the rectangles of K into several categories. After this categorization, the description of the unfolding layout is not difficult.

For $i = 0, 1, \ldots, n-1$, define the *i-band* to be the set of vertical rectangles (i.e., that lie in an xz plane or in a yz plane) whose z coordinates are between z_i and z_{i+1}. Each i-band is connected, and all of the rectangles of an i-band have the same extent in the z dimension, namely, $[z_i, z_{i+1}]$.

For $i = 0, 1, \ldots, n$, we define the *i-faces* to be the faces of K in the horizontal plane $z = z_i$. As we have defined them, an i-face has several properties. It may have the interior of K above or below it (but not both). The perimeter of the i-face has a nonempty intersection with the $(i-1)$-band, provided $i > 0$, and with the i-band, provided $i < n$. (If an i-face does not meet the $(i-1)$-band, it must be the bottom face of the polyhedron, and if it did not meet the i-band, it must be the top face of the polyhedron.) By the definition of an orthostack, the intersection between an i-face and each incident band is connected. That is, the perimeter of the i-face can be cut into two contiguous intervals such that each interval intersects solely the $(i-1)$-band or the i-band.

Also needed are the notions of the "begin rectangle" and "end rectangle" of the i-band. Choose the *0-band begin rectangle* to be an arbitrary rectangle of the 0-band. For $i \geq 0$, define the *i-band end rectangle* to be the rectangle of the i-band that is adjacent to the i-band begin rectangle in the counter-clockwise direction as viewed from $+z$. (Thus, the begin and end rectangles of the i-band are adjacent.) For $i \geq 1$, define the *i-connecting face* to be the i-face that shares an edge with the $(i-1)$-band end rectangle, if such a face exists. For $i \geq 1$, define the *i-band begin rectangle* to be one of the rectangles of the i-band that shares an edge with the i-connecting face, if it exists, or else the rectangle of the i-band that shares an edge with the $(i-1)$-band end rectangle. The *i-band interior rectangles* are rectangles of the i-band that are neither the begin rectangle nor the end rectangle.

Define the *i-connecting sequence* to be an edge-connected sequence of rectangles in the i-connecting face, if it exists, starting at the rectangle that shares an edge with the $(i-1)$-band end rectangle and ending at the rectangle that shares an edge with the i-band begin rectangle. This sequence is chosen to contain the fewest rectangles possible (a shortest path in the dual graph on the rectangles in the i-connecting face), in order to prevent the path from looping around an island and thereby isolating interior portions of the i-face. If the i-connecting face does not exist, the i-connecting sequence is the empty sequence. The rectangles in the i-connecting sequence are called *i-connecting rectangles*; all other rectangles of the i-face are called *normal rectangles*.

We now merge all normal rectangles with their normal neighbors in the x dimension. Call the resultant rectangular regions *über-rectangles*. Thus i-faces are partitioned into the i-connecting rectangles and the i-über-rectangles. All i-über-rectangles are connected to the perimeter of the i-face, and thus are edge-connected to the $(i-1)$-band or the i-band. Define an *i-up-über-rectangle* to be an über-rectangle that is incident to the i-band and an *i-down-über-rectangle* to be an über-rectangle that is incident to the $(i-1)$-band. If an über-rectangle is incident to both, we classify it arbitrarily.

Thus we have partitioned K into i-band begin rectangles, i-band end rectangles, i-band interior rectangles, i-up-über-rectangles, i-down-über-rectangles, and i-connecting rectangles. We now proceed to a description of the unfolding.

2.2 Unfolding Algorithm

Our unfolding of an orthostack consists of several components strung together at distinguished rectangles called *anchors*. Specifically, there are two types of components, i-main components and i-connecting components, both of which are anchored at two rectangles, a begin rectangle and an end rectangle. The *i-main component* consists of the entire i-band (the i-band begin rectangle, the i-band end rectangle, and the i-band interior rectangles), the $(i+1)$-down-über-rectangles, and the i-up-über-rectangles. The *i-connecting component* consists of the $(i-1)$-band end rectangle, the i-connecting rectangles (if any), and the i-band begin rectangle. It serves to connect the $(i-1)$-main component and the i-main component (at the $(i-1)$-band end rectangle and the i-band begin rectangle, respectively).

To ensure that components do not overlap each other, we enforce that the components are *anchored* in the following sense. A component is anchored at anchor rectangles R and S if, in the unfolded layout of the component, no rectangles are in the shaded region of Figure 2. More precisely, every rectangle is strictly right of R and left of S, or directly below R, or directly above S.

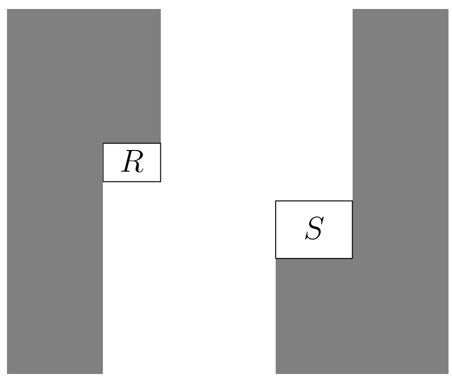

Fig. 2. If R and S are anchors of an anchored component, the component may be unfolded only in the unshaded region.

We can combine two anchored components with a common anchor while avoiding overlap. More precisely, given a component C anchored at anchors R and S, and another component C' anchored at S and T, we can combine the two unfolded layouts by rigidly moving C' so that the two copies of S coincide (with matching orientations). The conditions on the rectangles in the two components C and C' guarantees nonoverlap of the combined unfolded layout.

We unfold the orthostack in such a way that the positive z direction of every vertical (i-band) rectangle is placed in the positive y direction in the planar unfolding.

We edge-unfold the i-main component by leaving one edge attached between the über-rectangles of the component (arbitrarily, if there is a choice), and cutting along all of the other edges of the über-rectangles. As shown in Figure 3, the layout induced by this edge unfolding consists of a central horizontal rectangular strip, which contains all i-band rectangles, and has the $(i+1)$-down-

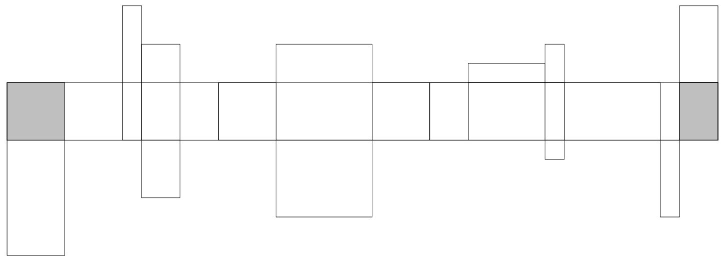

Fig. 3. An example of an unfolded i-main component. The shaded rectangles are the i-begin rectangle (right) and i-end rectangle (left). They are connected by the remainder of the i-band. Above the i-band are the $(i+1)$-down-über-rectangles and below are the i-up-über-rectangles.

über-rectangles connected to the top of this strip, and the i-up-über-rectangles connected to the bottom of this strip. The rightmost rectangle of this strip is the i-band begin rectangle, and the leftmost rectangle of the strip is the i-band end rectangle. There is nothing above the leftmost rectangle or below the rightmost rectangle because these vacant locations are where the connecting rectangles are attached, by definition, and we know that connecting rectangles are not über-rectangles. Therefore the edge unfolding of the i-main component is anchored at the i-band begin and end rectangles.

We vertex-unfold the i-connecting component by an incremental algorithm, by performing a sequence of vertex unfoldings, beginning with the $(i-1)$-band end rectangle. This unfolding proceeds in phases: the middle phase and the end phase.

The middle phase of the vertex unfolding does all unfoldings except for the last (i.e., it is used for all unfoldings that do not involve the i-band begin rectangle). See Figure 4. This phase has the property that every rectangle is to the left of the previous rectangle. Suppose we are unfolding R and S, and the next item to be unfolded is T. Our algorithm requires as a precondition that S not be to the right of R and as a postcondition ensures that T is not to the right of S. The unfolding begins with the i-band end rectangle and the first rectangle of the i-connecting sequence. In order to place the i-band begin rectangle with the proper orientation, the first rectangle of the i-connecting sequence must be adjacent to its top edge. Thus, it satisfies the precondition for this construction. The construction has three cases, depending on whether S is above, below, or to the left of R. In the first two cases, we vertex-unfold $90°$ about the leftmost point of the edge conneting R and S so that S is to the left of R, while in the third case we do nothing.

The end phase is trickier because the i-begin rectangle must be oriented properly. If the i-connecting face does not exist, the i-begin rectangle is connected to the top edge of the $(i-1)$-end rectangle, and we are done. Otherwise, because of the construction in the middle phase, the i-begin rectangle may be connected to the top, left, or bottom edge of the last rectangle in the i-connecting sequence.

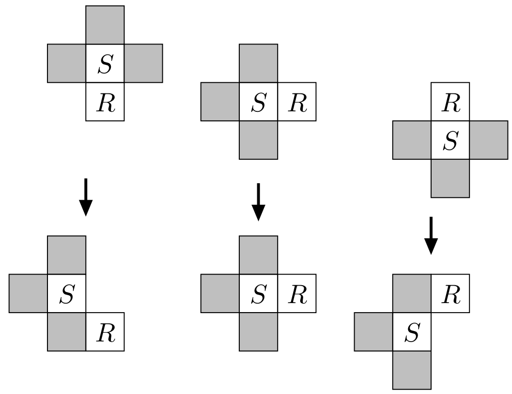

Fig. 4. How a path of rectangles can be vertex-unfolded so that each rectangle is to the left of the previous rectangle. In this diagram, R represents the current rectangle, S represents the next rectangle, and the grey shaded area represent the three possible positions of the next next rectangle. There are three cases of the possible location of S in relation to R. Note that the illustrated unfoldings work no matter what the sizes of the rectangles.

The i-begin rectangle must be oriented so that the edge that connected it to the last i-connecting rectangle is the bottom edge when unfolded. There are three cases, illustrated in Figure 5.

Thus, the i-connecting component can always be vertex-unfolded into an anchored unfolding. Because the main component can also be vertex-unfolded into an anchored component, we conclude

Theorem 1. *Every orthostack can be grid vertex-unfolded.*

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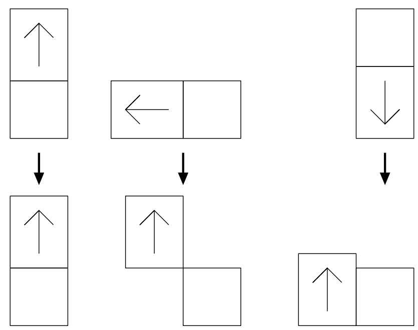

Fig. 5. How the end phase unfolds the connecting component. The empty rectangle represents the last i-connecting rectangle, and the rectangle with the arrow represents the i-begin rectangle. There are three cases for their configuration before unfolding; after unfolding the arrow must point up in order for this rectangle to be in the same orientation as in the main component. This figure shows how this is done. Note that the illustrated unfoldings work no matter what the sizes of the rectangles.

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