Reaching Folded States of a Rectangular Piece of Paper

Erik D. Demaine* Joseph S. B. Mitchell†

Abstract

We prove that any folded state of a rectangular piece of paper (a continuous isometric non-self-intersecting mapping of the paper into space) can be reached by a continuous folding process, starting from the unfolded state, while at all times being a valid folding. In our model, the paper cannot properly cross itself, but can touch itself, as in a flat folding.

1 Introduction

Inspired by discussions with Jack Edmonds (August 1997) and with Marshall Bern (1998), Erik Demaine, Martin Demaine, and Anna Lubiw (1998) posed the following paper-folding problem: Can every folded state be folded?

Formally, a *piece of paper* is a polygonal region, possibly with holes. More generally, we can consider the surfaces of polyhedra as pieces of paper. A *folded state* of a piece of paper is a function mapping it into $\mathbf{R}^3$ that satisfies three properties:

1. it is continuous (no ripping),
2. it is isometric, i.e., preserves distances along the surface (no stretching), and
3. it does not properly cross itself, although it may touch itself.

The last property is best visualized by thinking of the folded state as having infinitesimal separations between the layers of paper that are mapped to a common surface, thereby defining an ordering among the portions of the paper that come in contact with each other. A folded state is *polyhedral* if it is piecewise-affine, i.e., it is the union of a finite number of triangles. In particular, all *flat-folded states*—folded states that lie entirely within a plane—are polyhedral.

A *folding process* is a continuous motion that is at all times a folded state. In other words, a folding process of a piece of paper P is a function $f : P \times [0, 1] \to \mathbf{R}^3$ such that each snapshot $f(\cdot, t) : P \to \mathbf{R}^3$, $t \in [0, 1]$, is a folded state. Here the unit interval $[0, 1]$ denotes time. The final snapshot $f(\cdot, 1)$ is called the *result* of the folding process. We normally assume that the initial snapshot $f(\cdot, 0)$ is the *unfolded state*: a rigid copy of the piece of paper P in $\mathbf{R}^3$.

Now the open problem we address here is this: Is every folded state the result of a folding process starting from the unfolded state? The Demaine-Demaine-Lubiw conjecture is that the answer is yes, for flat polygonal pieces of paper as well as for polyhedral surfaces.

We settle this question in the positive for (flat) rectangular pieces of paper and polyhedral folded states, which is the primary case of interest in origami. In fact, we need only a much weaker condition than polyhedral, namely, that a small portion of the folded state is polyhedral:

Theorem 1 *Any folded state of a rectangular piece of paper, having a positive-area region that is flat (affine), can be reached by a folding process.*

Our proof is constructive. For a polyhedral folded state, our folding process has a finite number of creases at any moment in time, but an uncountably infinite number of creases overall.

1.1 Consequences

Our result shows that, for rectangular pieces of paper and polyhedral folded states, the existence of a folded state is equivalent to the existence of a folding process. This implies that other results can focus on finding folded states instead of folding processes. In particular, several results about folded states generalize to folding processes. For example:

- Deciding whether a crease pattern can be continuously folded into a flat origami is NP-complete [3].
- Any rectangle can be continuously folded in order to line up all the edges of a specified planar graph and no more [2, 4].
- Uniaxial bases forming a desired metric tree can be continuously folded [5].

*Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada, email: eddemaine@uwaterloo.ca.

†Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, NY 11794-3600, USA, email: jsbm@ams.sunysb.edu.

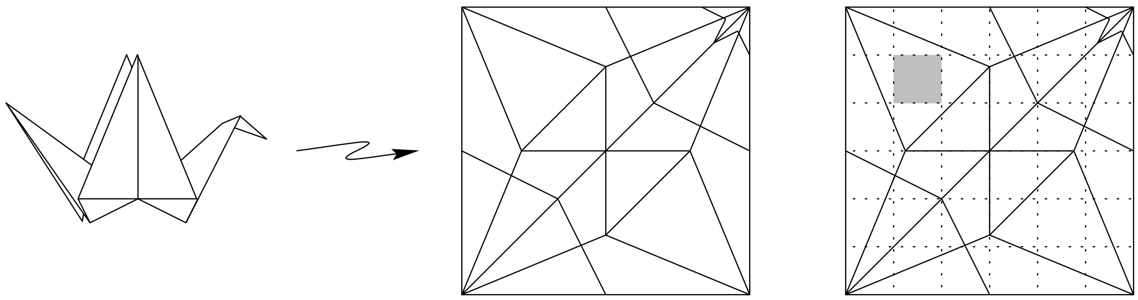

Figure 1: (Left) Crease pattern for an origami crane. (Right) Division into a regular grid so that one cell (shaded) contains no creases.

2 Solution

By invariance under rigid motion, we assume that the rectangular piece of paper P is in the xy plane and axis-aligned, its span in the x dimension is $[0, X]$, and its span in the y dimension in $[0, Y]$. Let S denote the target folded state, so $S : P \to \mathbf{R}^3$.

We begin by subdividing the piece of paper P (as in Figure 1) into a sufficiently fine regular $(1/\varepsilon_x) \times (1/\varepsilon_y)$ grid, so that one cell E remains unfolded by S, i.e., such that S is affine on E. In particular, for polyhedral foldings, E is an axis-aligned rectangle completely contained in a face of the crease pattern. (In general, we are using that at least some positive-area portion of the folded state is flat.) Furthermore, if $[o_x, o_x + \varepsilon_x] \times [o_y, o_y + \varepsilon_y]$ are the coordinates of E, then o_x and X are both integer multiples of ε_x, and similarly for y.

We will use this unfolded subrectangle E, or more precisely the mapped copy $S(E)$, as a "dock" for "unrolling" the piece of paper into the folded state. More precisely, the folding process of P into S is split into five steps, detailed in the next sections:

1. folding the piece of paper P into a thin vertical strip $\widehat{V}$,[1]
2. folding that strip $\widehat{V}$ into a small rectangle $\widehat{E}$,
3. moving the small rectangle $\widehat{E}$ into place, at $S(E)$,
4. unrolling the small rectangle $\widehat{S(E)}$ into an appropriately contorted strip $\widehat{S(V)}$, and
5. unrolling the strip $\widehat{S(V)}$ into the desired folded state S.

Step 1: Folding the Piece of Paper P into a Thin Vertical Strip $\widehat{V}$. If we extend the subrectangle E vertically so as to span all cells in the same column, we obtain a vertical $\varepsilon_x \times Y$ strip, V, contained in P. We begin by folding the piece of paper P down to a multilayer version of this strip, $\widehat{V}$, as in Figure 2. In this folding, the creases are parallel to the y axis, and alternate between mountains and valleys. One way to achieve this folding by a continuous motion is as a sequence of "simple folds" [1], each fold performing a rotation by $\pm 180°$ along a single line. The result is an $\varepsilon_x \times Y$ accordion, $\widehat{V}$, occupying the same space as the desired strip V.

[1] The $\widehat{\ }$ notation indicates that there are multiple layers in addition to those in the unfolded state P or the folded state S.

Step 2: Folding the Thin Strip $\widehat{V}$ into a Small Rectangle $\widehat{E}$. Now our target is a multilayer version of the small subrectangle E of P. To achieve this we simply repeat Step 1, except that the fold lines are now horizontal. Now we have folded the piece of paper into a small rectangle, $\widehat{E}$, occupying the same space as E.

Step 3: Moving the Small Rectangle $\widehat{E}$ into Place, at $S(E)$. Currently our small folded rectangle $\widehat{E}$ is located appropriately for the unfolded state P, i.e., on top of E. By applying a rigid motion, we can reposition it to where it belongs in the folded state S, i.e., on top of $S(E)$.

Step 4: Unrolling the Small Rectangle $\widehat{E}$ into a Folded Strip $\widehat{V}$. The last two steps are the most interesting. The idea is to "unroll" the folded-up rectangle by a continuous sliding process, while simultaneously matching the desired folded state S. From now on, the folding will be a subset of the final folded state; the portion of the folded state achieved will grow over time.

We continuously "pull" on a horizontal edge of the small rectangle, as shown in Figure 3, gradually reducing the number of layers and unrolling into the vertical strip V. We use this motion to continuously "feed" material along the folded state, to form $S(V)$. More precisely, if we examine a horizontal cross-section of the folded strip, then as the cross-section leaves the small rectangle $S(E)$, it follows precisely along the surface

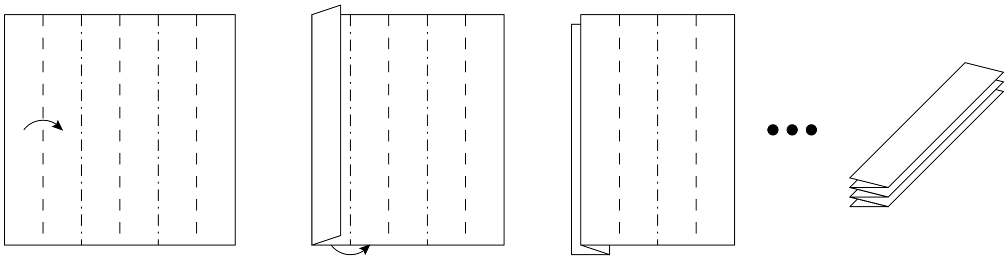

Figure 2: "Accordion folding" a rectangle into a strip.

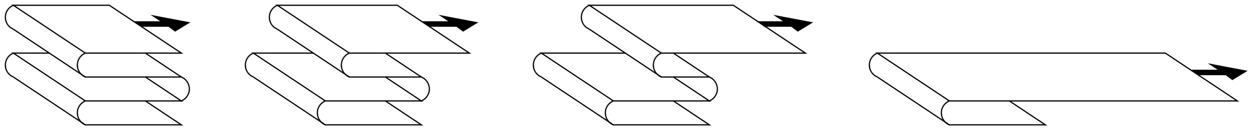

Figure 3: Unrolling the small rectangle back into a thin strip by a continuous "feeding" process. In this illustration, the infinitesimal distances separating the layers of the small rectangle have been expanded to visible values, whereas the layers of the strip are not separated. In the last frame, the number of layers in the small rectangle has decreased.

$S(V)$. That is, the horizontal cross-section is defined to be S applied to horizontal cross-sections of V.

We need to apply this unrolling motion in potentially two directions, to extend in both $-y$ and $+y$. More precisely, o_y/ε_y units (layers of $\widehat{E}$) are pulled in the $-y$ direction, and the remaining units are pulled in the $+y$ direction. The resulting folding, $\widehat{S(V)}$, occupies the same space as one strip of the folded state, $S(V)$.

Step 5: Unrolling the Strip $\widehat{S(V)}$ into the Desired Folded State S. Finally, we fold the contorted strip $\widehat{S(V)}$ by the same motions as in Step 4, except in the x dimension. The result is precisely the desired folded state S, having reduced to a single layer of paper (other than the layering in S).

3 Conclusion

There are two main open problems:

1. What about other pieces of paper, either general polygons or surfaces of polyhedra?
2. When can such foldings use a finite number of creases overall? We conjecture that this is possible for a piece of paper with zero curvature at every point (e.g., a planar polygon). Connelly has shown that this is impossible for pieces of paper with points of positive curvature.

Another more technical question is whether we can remove the assumption in Theorem 1 about a portion of the folding being polyhedral.

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