Approximation Schemes for Planar Graph Problems (1983, 1994; Baker)

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1 PROBLEM DEFINITION

Many NP-hard graph problems become easier to approximate on planar graphs and their generalizations. (A graph is *planar* if it can be drawn in the plane (or the sphere) without crossings. For definitions of other related graph classes, see the entry on Bidimensionality (2004; Demaine, Fomin, Hajiaghayi, Thilikos).) For example, *maximum independent set* asks to find a maximum subset of vertices in a graph that induce no edges. This problem is inapproximable in general graphs within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$ unless NP = ZPP (and inapproximable within $n^{1/2-\epsilon}$ unless P = NP), while for planar graphs there is a 4-approximation (or simple 5-approximation) by taking the largest color class in a vertex 4-coloring (or 5-coloring). Another is *minimum dominating set*, where the goal is to find a minimum subset of vertices such that every vertex is either in or adjacent to the subset. This problem is inapproximable in general graphs within $\epsilon \log n$ for some $\epsilon > 0$ unless P = NP, but as we will see, for planar graphs the problem admits a *polynomial-time approximation scheme* (PTAS): a collection of $(1+\epsilon)$-approximation algorithms for all $\epsilon > 0$.

There are two main general approaches for designing PTASs for problems on planar graphs and their generalizations: the separator approach and the Baker approach.

Lipton and Tarjan [15, 16] introduced the first approach, which is based on planar separators. The first step in this approach is to find a separator of $O(\sqrt{n})$ vertices or edges, where n is the size of the graph, whose removal splits the graph into two or more pieces each of which is a constant fraction smaller than the original graph. Then recurse in each piece, building a recursion tree of separators, and stop when the pieces have some constant size such as $1/\epsilon$. The problem can be solved on these pieces by brute force, and then it remains to combine the solutions up the recursion tree. The induced error can often be bounded in terms of the total size of all separators, which in turn can be bounded by ϵn. If the optimal solution is at least some constant factor times n, this approach often leads to a PTAS.

There are two limitations to this planar-separator approach. First, it requires that the optimal solution is at least some constant factor times n; otherwise, the cost incurred by the separators can be far larger than the desired optimal solution. Such a bound is possible in some problems after some graph pruning (linear kernelization), e.g., independent set, vertex cover, and forms of the Traveling Salesman Problem. But, for example, Grohe [12] states that dominating set is a problem "to which the technique based on the separator theorem does not apply". Second, the approximation algorithms resulting from planar separators are often impractical because of large constant factors. For example, to achieve an approximation ratio of just 2, the base case requires exhaustive solution of graphs of up to $2^{2^{400}}$ vertices.

Baker [1] introduced her approach to address the second limitation, but it also addresses the first limitation to a certain extent. This approach is based on decomposition into overlapping subgraphs of bounded outerplanarity, as described in the next section.

2 KEY RESULTS

Baker's original result [1] is a PTAS for maximum independent set (as defined above) on planar graphs, as well as the following list of problems on planar graphs: maximum tile salvage, partition into triangles, maximum H-matching, minimum vertex cover, minimum dominating set, and minimum edge-dominating set.

Baker's approach starts with a planar embedding of the planar graph. Then it divides vertices into *layers* by iteratively removing vertices on the outer face of the graph: layer j consists of the vertices removed at the jth iteration. If one now removes the layers congruent to i modulo k, for any choice of i, the graph separates into connected components each with at most k consecutive layers, and hence the graph becomes k-outerplanar. Many NP-complete problems can be solved on k-outerplanar graphs for fixed k using dynamic programming (in particular, such graphs have bounded treewidth). Baker's approximation algorithm computes these optimal solutions for each choice i of the congruence class of layers to remove, and returns the best solution among these k solutions. The key argument for maximization problems considers the optimal solution to the full graph, and argues that the removal of one of the k congruence classes of layers must remove at most a $1/k$ fraction of the optimal solution, so the returned solution must be within a $1 + 1/k$ factor of optimal. A more delicate argument handles minimization problems as well. For many problems, such as maximum independent set, minimum dominating set, and minimum vertex cover, Baker's approach obtains a $(1 + \epsilon)$-approximation algorithms with running time of $2^{O(1/\epsilon)}n^{O(1)}$ on planar graphs.

Eppstein [10] generalized Baker's approach to a broader class of graphs called graphs of *bounded local treewidth*, i.e., where the treewidth of the subgraph induced by the set of vertices at distance at most r from any vertex is bounded above by some function $f(r)$ independent of n. The main differences in Eppstein's approach are replacing the concept of bounded outerplanarity with the concept of bounded treewidth, where dynamic programming can still solve many problems, and labeling layers according to a simple breadth-first search. This approach has led to PTASs for hereditary maximization problems such as maximum independent set and maximum clique, maximum triangle matching, maximum H-matching, maximum tile salvage, minimum vertex cover, minimum dominating set, minimum edge-dominating set, minimum color sum, and subgraph isomorphism for a fixed pattern [10, 8, 6]. Frick and Grohe [11] also developed a general framework for deciding any property expressible in first-order logic in graphs of bounded local treewidth.

The foundation of these results is Eppstein's characterization of minor-closed families of graphs with bounded local treewidth [10]. Specifically, he showed that a minor-closed family has bounded local treewidth if and only if it excludes some *apex graph*, a graph with a vertex whose removal leaves a planar graph. Unfortunately, the initial proof of this result brought Eppstein's approach back to the realm of impracticality, because his bound on local treewidth in a general apex-minor-free graph is doubly exponential in r: $2^{2^{O(r)}}$. Fortunately, this bound could be improved to $2^{O(r)}$ [3] and even the optimal $O(r)$ [4]. The latter bound restores Baker's $2^{O(1/\epsilon)}n^{O(1)}$ running time for $(1+\epsilon)$-approximation algorithms, now for all apex-minor-free graphs.

Another way to view the necessary decomposition of Baker's and Eppstein's approaches is that the vertices or edges of the graph can be split into any number k of pieces such that deleting any one of the pieces results in a graph of bounded treewidth (where the bound depends on k). Such decompositions in fact exist for arbitrary graphs excluding any fixed minor H [9], and they can be found in polynomial time [6]. This approach generalizes the Baker-Eppstein PTASs described above to handle general H-minor-free graphs.

This decomposition approach is effectively limited to *deletion-closed* problems, whose optimal solution only improves when deleting edges or vertices from the graph. Another decomposition approach targets *contraction-closed* problems, whose optimal solution only improves when contracting edges. These problems include classic problems such as dominating set and its variations, the Traveling Salesman Problem, subset TSP, minimum Steiner tree, and minimum-weight c-edge-connected submultigraph. PTASs have been obtained for these problems in planar graphs [13, 14, 2] and in bounded-genus graphs [7] by showing that the edges can be decomposed into any number k of pieces such that contracting any one piece results in a bounded-treewidth graph (where the bound depends on k).

3 APPLICATIONS

Most applications of Baker's approach have been limited to optimization problems arising from "local" properties (such as those definable in first-order logic). Intuitively, such local properties can be decided by locally checking every constant-size neighborhood. In [5], Baker's approach is generalized to obtain PTASs for nonlocal problems, in particular, connected dominating set. This generalization requires the use of two different techniques. The first technique is to use an ϵ-fraction of a constant-factor (or even logarithmic-factor) approximation to the problem as a "backbone" for achieving the needed nonlocal property. The second technique is to use subproblems that overlap by $\Theta(\log n)$ layers instead of the usual $\Theta(1)$ in Baker's approach.

Despite this advance in applying Baker's approach to more general problems, the planar-separator approach can still handle some different problems. Recall, though, that the planar-separator approach was limited to problems in which the optimal solution is at least some constant factor times n. This limitation has been overcome for a wide range of problems [5], in particular obtaining a PTAS for feedback vertex set, to which neither Baker's approach nor the planar-separator approach could previously apply. This result is based on evenly dividing the optimum solution instead of the whole graph, using a relation between treewidth and the optimal solution value to bound the treewidth of the graph, and thus obtaining an $O(\sqrt{\mathrm{OPT}})$ separator instead of an $O(\sqrt{n})$ separator. The $O(\sqrt{\mathrm{OPT}})$ bound on treewidth follows from the bidimensionality theory described in the entry on Bidimensionality (2004; Demaine, Fomin, Hajiaghayi, Thilikos). We can divide the optimum solution into roughly even pieces, without knowing the optimum solution, by using existing constant-factor (or even logarithmic-factor) approximations for the problem. At the base of the recursion, pieces no longer have bounded size but do have bounded treewidth, so fast fixed-parameter algorithms can be used to construct optimal solutions.

4 OPEN PROBLEMS

An intriguing direction for future research is to build a general theory for PTASs of subset problems. Although PTASs for subset TSP and Steiner tree have recently been obtained for planar graphs [14, 2], there remain several open problems of this kind, such as subset feedback vertex set.

Another instructive problem is to understand the extent to which Baker's approach can be applied to nonlocal problems. Again there is an example of how to modify the approach to handle the nonlocal problem of connected dominating set [5], but for example the only known PTAS for feedback vertex set in planar graphs follows the separator approach.

5 CROSS REFERENCES

- Bidimensionality (2004; Demaine, Fomin, Hajiaghayi, Thilikos)
- Separators (1988, 1999; Leighton, Rao)
- Treewidth of Graphs (2005; Bodlaender)

6 RECOMMENDED READING

[1] Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the Association for Computing Machinery*, 41(1):153–180, 1994.

[2] Glencora Borradaile, Claire Kenyon-Mathieu, and Philip N. Klein. A polynomial-time approximation scheme for Steiner tree in planar graphs. In *Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2007.

[3] Erik D. Demaine and MohammadTaghi Hajiaghayi. Diameter and treewidth in minor-closed graph families, revisited. *Algorithmica*, 40(3):211–215, August 2004.

[4] Erik D. Demaine and MohammadTaghi Hajiaghayi. Equivalence of local treewidth and linear local treewidth and its algorithmic applications. In *Proceedings of the 15th ACM-SIAM Symposium on Discrete Algorithms (SODA'04)*, pages 833–842, January 2004.

[5] Erik D. Demaine and MohammadTaghi Hajiaghayi. Bidimensionality: New connections between FPT algorithms and PTASs. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005)*, pages 590–601, Vancouver, January 2005.

[6] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: Decomposition, approximation, and coloring. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 637–646, Pittsburgh, PA, October 2005.

[7] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Bojan Mohar. Approximation algorithms via contraction decomposition. In *Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 278–287, New Orleans, Louisiana, January 7–9 2007.

[8] Erik D. Demaine, MohammadTaghi Hajiaghayi, Naomi Nishimura, Prabhakar Ragde, and Dimitrios M. Thilikos. Approximation algorithms for classes of graphs excluding single-crossing graphs as minors. *Journal of Computer and System Sciences*, 69(2):166–195, September 2004.

[9] Matt DeVos, Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, Bruce Reed, Paul Seymour, and Dirk Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *Journal of Combinatorial Theory, Series B*, 91(1):25–41, 2004.

[10] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000.

[11] Markus Frick and Martin Grohe. Deciding first-order properties of locally tree-decomposable structures. *Journal of the ACM*, 48(6):1184–1206, 2001.

[12] Martin Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.

[13] Philip N. Klein. A linear-time approximation scheme for TSP for planar weighted graphs. In *Proceedings of the 46th IEEE Symposium on Foundations of Computer Science*, pages 146–155, 2005.

[14] Philip N. Klein. A subset spanner for planar graphs, with application to subset TSP. In *Proceedings of the 38th ACM Symposium on Theory of Computing*, pages 749–756, 2006.

[15] Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[16] Richard J. Lipton and Robert Endre Tarjan. Applications of a planar separator theorem. *SIAM Journal on Computing*, 9(3):615–627, 1980.