Open Problems on Polytope Reconstruction

Erik D. Demaine* Jeff Erickson†

Abstract. We describe some open algorithmic problems related to constructing 3-dimensional polytopes from limited information.

Introduction. There are several different ways to specify convex polytopes in three dimensions. One obvious explicit representation is a list of the vertex coordinates and connectivity information between the vertices, edges, and facets. But not all this information is necessary. For example, we can reconstruct the explicit representation given only the vertex coordinates; this is the well-studied *convex-hull* problem. By projective duality, we can also reconstruct the polytope from a list of halfspaces whose intersection is the polytope.

Although these are the two most well-known ways to specify polytopes, at least in the computational geometry community, they are not the only ones. Here we list several different theorems describing surprisingly small sets of information that are sufficient to specify 3-dimensional convex polytopes. The purpose of this note is to pose the algorithmic versions of these theorems as intriguing open questions. That is, is there an algorithm that, given the information uniquely specifying a convex polytope, builds the polytope? Numerical approximation algorithms are known in some cases, but we know of no exact, purely combinatorial algorithms. Such algorithms may require a model of computation that allows exact computation (or at least useful representations) of high-degree algebraic numbers.

Nets ("unfoldings"). Our problems were inspired by one in particular, Aleksandrov's theorem, which we detail now. A *polyhedral metric* on the sphere assigns to each point a neighborhood that is isometric to an open planar disk, except for a finite number of points whose neighborhoods are isometric to the apex of a cone. If the complete angle around every cone point is at most 2π, the metric is said to be *convex*. Equivalently, a polyhedral metric is obtained by gluing together edges of a simple polygon in equal-length pairs, so that the resulting 2-complex is homeomorphic to a sphere; the metric is convex if the sum of the angles incident to each vertex is at most 2π. This glued simple polygon is called a *net* (see Figure 1 for an example).

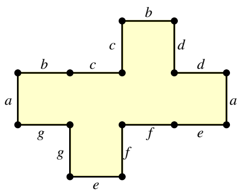

Figure 1. A net for the cube.

Any convex 3-polytope naturally defines a polyhedral metric; the "distance" between two points is just the length of the shortest path on the polytope's surface. A beautiful and surprising result of Aleksandrov is that the converse is true as well.

Aleksandrov's Theorem. [1] *Any convex polyhedral metric can be realized by a unique convex polytope (up to congruence).*

For any convex polytope, we can define a net by "unfolding" it into the plane. It is a well-known open problem [3, 7, 8] whether every polytope can be unfolded into a simple net, that is, one that does not overlap itself, by cutting along edges; however, even non-simple nets can be used to define polyhedral metrics. Aleksandrov's theorem states that any net is an unfolding of a unique convex polytope (up to congruence).

Problem 1. [5, 7] *Given a net (a simple polygon with edge-matching rules), how quickly can we construct the corresponding polytope?*

O'Rourke [7] describes how to split this into two separate subproblems. The first is to find the *creases* on the polygon which map to polytope edges. A superset of these creases can be found in polynomial time [7], but it remains open to isolate them exactly.

Once we've found (a superset of) the creases, we have the shapes of the faces and the complete adjacency information—which faces are adjacent along which edges. If two polytopes have the same faces and the same adjacency pattern, they are called *stereoisomers*. A key step in the proof of the uniqueness part of Aleksandrov's theorem is the following beautiful result of Cauchy:

Cauchy's Rigidity Theorem. *Convex stereoisomers are congruent.*

In other words, any set of polygons with adjacency information can be obtained from at most one polytope, and (by the existence part of Aleksandrov's theorem) this polytope exists as long as the resulting complex is homeomorphic to the sphere and the sum of the angles around each vertex is at most 2π. How quickly can we construct this polytope? Cauchy's proof is nonconstructive. Although a numerical-approximation algorithm seems quite likely, and indeed such experiments have been carried out [7], it would be much more interesting to have a purely combinatorial algorithm that runs in polynomial time.

Here are two related open questions, which we conjecture to be NP-hard.

Problem 2. *Given a set of convex polygons* without *adjacency information, how quickly can we decide whether they can be assembled into a convex polytope? Into a unique convex polytope?*

*Dept. of Computer Science, Univ. of Waterloo, Waterloo, eddemain@uwaterloo.ca; http://daisy.uwaterloo.ca/~eddemain/

†Dept. of Computer Science, Univ. of Illinois, jeffe@cs.uiuc.edu; http://www.uiuc.edu/~jeffe/

Problem 3. *How quickly can we actually construct a polytope with a given set of facets, if one exists?*

Area-weighted normal vectors. A fairly simple theorem of Minkowski states that if you take the normal vectors of the facets of a convex polytope, where the length of the vector is the area of the corresponding facet, then the resulting collection of vectors sum to zero. In fact, this theorem is true in any dimension, where "area" means the natural $(d-1)$-dimensional Lebesgue measure. But the more interesting part of Minkowski's theorem is that this process is reversible.

Minkowski's Theorem. *Any set of vectors whose sum is zero is the set of area-weighted normals of a unique convex polytope (up to translation).*

Problem 4. *Given a set of vectors that sum to zero, how quickly can we construct the corresponding polytope?*

Aurenhammer, Hoffmann, and Aronov [2] describe an iterative numerical-approximation algorithm for this problem, by recasting it as a convex-optimization problem. However, no purely combinatorial algorithm is known.

Several "easier" decision problems are also open. For example, given a set of vectors whose sum is zero, is the corresponding polytope simple (every vertex has degree three)? Is it simplicial (every facet is a triangle)? Is its volume less than 1?

1-skeleta ("Edge Graphs"). The 1-*skeleton* of a convex polytope is the graph naturally induced by the polytope's vertices and edges. The following well-known result of Steinitz completely characterizes the 1-skeleta of convex 3-polytopes:

Steinitz's Theorem. [10] *A graph is the* 1*-skeleton of a (not necessarily unique) convex* 3*-polytope if and only if it is planar and* 3*-connected.*

Problem 5. *Given a* 3*-connected planar graph, how quickly can we construct a polytope whose* 1*-skeleton is that graph?*

Das and Goodrich [4] describe an algorithm to realize any 3-connected planar *triangulation* as a polytope in $O(n)$ time on a rational RAM, but the problem remains open for non-triangulated graphs.

One way to construct such a polytope might be to use the following theorem of Koebe, independently reproved by Thurston using results of Andreev.

Koebe's Theorem. [10] *Any planar graph is the contact graph of a set of circular disks in the plane or circular caps on the sphere. Furthermore, if the graph is a triangulation, the set of disks is unique up to Möbius transformations.*

Koebe's theorem can be used to prove the following much stronger version of Steinitz's theorem, which describes a "canonical" polytope representation of a graph. (The history of this theorem is a bit muddled [10]; overlapping portions were independently proved—but not necessarily published—by Brägger, Doyle, Schramm, and Thurston.)

Theorem. *Any* 3*-connected planar graph is the* 1*-skeleton of a polytope with edges tangent to the unit sphere, such that the barycenter of the contact points is the origin. This polytope is unique up to reflections and rotations about the origin, and every combinatorial symmetry of the graph is realized by a symmetry of the polytope. Each edge of the polytope meets the corresponding edge of the polar dual polytope at right angles at the contact point on the sphere.*

Problem 6. *Given a* 3*-connected planar graph, how quickly can we construct its "canonical" polytope?*

This problem is open even for triangulations; Das and Goodrich's algorithm [4] contructs non-canonical polytopes. Solving this problem essentially boils down to finding an algorithmic version of Koebe's theorem.

Problem 7. *How quickly can we construct a set of disks with a given planar contact graph?*

The proofs of Koebe, Andreev, and Thurston are nonconstructive, as are several more recent proofs. Mohar [6] and Smith [9] independently developed polynomial-time numerical-approximation algorithms. No purely combinatorial algorithm is known, which is perhaps not surprising since the radii could be algebraic numbers of unbounded degree.

We conclude with another related open question.

Problem 8. *How hard is it to decide if a collection of "sticks" (line segments subject to rigid motions) can be joined to form the* 1*-skeleton of a convex polytope?*

The corresponding two-dimensional question has an easy answer: A set of sticks can be assembled into a convex polygon if and only if the longest stick is shorter than all the other sticks put together. Like Problem 2, we conjecture that this problem is NP-hard.

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