Refold Rigidity of Convex Polyhedra

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Abstract

We show that every convex polyhedron may be unfolded to one planar piece, and then refolded to a different convex polyhedron. If the unfolding is restricted to cut only edges of the polyhedron, then we show that many regular and semi-regular polyhedra are "edge-unfold rigid" in the sense that each of their unfoldings may only fold back to the original. For example, all of the 43,380 edge unfoldings of a dodecahedron may only fold back to the dodecahedron. We begin the exploration of which polyhedra are edge-unfold rigid, demonstrating infinite rigid classes through perturbations, and identifying one infinite nonrigid class: tetrahedra.

(The full version of this paper is available.[1])

1 Introduction

It has been known since [5] and [3] that there are convex polyhedra, each of which may be unfolded to a planar polygon and then refolded to different convex polyhedra. For example, the cube may be unfolded to a "Latin cross" polygon, which may be refolded to 22 distinct non-cube convex polyhedra [4, Figs. 25.32-6]. But there has been only sporadic progress on understanding which pairs of convex polyhedra[2] have a common unfolding. A notable recent exception is the discovery [7] of an unfolding of a cube that refolds to a regular tetrahedron, partially answering Open Problem 25.6 in [4, p. 424].

Here we begin to explore a new question, which we hope will shed light on the unfold-refold spectrum of problems: Which polyhedra $\mathcal{P}$ are *refold-rigid* in the sense that any unfolding of $\mathcal{P}$ may only be refolded back to $\mathcal{P}$? The answer we provide here is: NONE— Every polyhedron $\mathcal{P}$ has an unfolding that refolds to an incongruent $\mathcal{P}'$. Thus every $\mathcal{P}$ may be *transformed* to some $\mathcal{P}'$.

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[1] http://cs.smith.edu/~orourke/Papers/RefoldRigidEuroCGFull.pdf.

[2] All polyhedra in this paper are convex, and the modifier will henceforth be dropped.

This somewhat surprising answer leads to the next natural question: Suppose the unfoldings are restricted to *edge unfoldings*, those that only cut along edges of $\mathcal{P}$ (rather than permitting arbitrary cuts through the interior of faces). Say that a polyhedron $\mathcal{P}$ whose every edge unfolding only refolds back to $\mathcal{P}$ is *edge-unfold rigid*, and otherwise is an *edge-unfold transformer*. It was known that four of the five Platonic solids are edge-unfold transformers (e.g., [2] and [6]). Here we prove that the dodecahedron is edge-unfold rigid: all of its edge unfoldings only fold back to the dodecahedron. The proof also demonstrates edge-unfold rigidity for 11 of the Archimedean solids. We also establish the same rigidity for infinite classes of slightly perturbed versions of these polyhedra. In contrast to this, we show that every tetrahedron is an edge-unfold transformer: at least one among a tetrahedron's 16 edge unfoldings refolds to a different polyhedron.

This work raises many new questions, summarized in Section 6.

2 Notation and Definitions

We will use $\mathcal{P}$ for a polyhedron in $\mathbb{R}^3$ and P for a planar polygon. An *unfolding* of a polyhedron $\mathcal{P}$ is development of its surface after cutting to a single (possibly overlapping) polygon P in the plane. The surface of $\mathcal{P}$ must be cut open by a spanning tree to achieve this. An *edge-unfolding* only includes edges of $\mathcal{P}$ in its spanning cut tree. Note that we do not insist that unfoldings avoid overlap.

A *folding* of a polygon P is an identification of its boundary points that satisfies the three conditions of Alexandrov's theorem: (1) The identifications (or "gluings") close up the perimeter of P without gaps; (2) The resulting surface is homeomorphic to a sphere; and (3) Identifications result in $\leq 2\pi$ angle glued at every point. Under these three conditions, Alexandrov's theorem guarantees that the folding produces a convex polyhedron, unique once the gluing is specified. See [1] or [4]. Note that there is no restriction that whole edges of P must be identified to whole edges, even when P is produced by an edge unfolding. We call a gluing that satisfies the above conditions an *Alexandrov gluing*.

A polyhedron $\mathcal{P}$ is *refold-rigid* if every unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$. Otherwise, $\mathcal{P}$ is a *transformer*. A polyhedron is *edge-unfold rigid* if every edge unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$, and otherwise it is an *edge-unfold transformer*.

3 Polyhedra Are Transformers

The proof that no polyhedron $\mathcal{P}$ is refold rigid breaks naturally into two cases. We first state a lemma that provides the case partition. Let $\kappa(v)$ be the curvature at vertex $v \in \mathcal{P}$, i.e., the "angle gap" at v: 2π minus the total incident face angle $\alpha(v)$ at v. By the Gauss-Bonnet theorem, the sum of all vertex curvatures of $\mathcal{P}$ is 4π.

Lemma 1 *For every polyhedron $\mathcal{P}$, either there is a pair of vertices with $\kappa(a) + \kappa(b) > 2\pi$, or there are two vertices each with at most π curvature: $\kappa(a) \le \pi$ and $\kappa(b) \le \pi$.*

Proof. Suppose there is no pair with curvature sum more than 2π. So we have $\kappa(v_1) + \kappa(v_2) \le 2\pi$ and $\kappa(v_3) + \kappa(v_4) \le 2\pi$ for four distinct vertices. Suppose neither of these pairs have both vertices with at most π curvature. If $\kappa(v_2) > \pi$, then $\kappa(v_1) \le \pi$; and similarly, if $\kappa(v_4) > \pi$, then $\kappa(v_3) \le \pi$. Thus we have identified two vertices, v_1 and v_3, both with at most π curvature. □

We can extend this lemma to accommodate 3-vertex doubly covered triangles as polyhedra, because then every vertex has curvature greater than π.

Lemma 2 *Any polyhedron $\mathcal{P}$ with a pair of vertices with curvature sum more than 2π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*

Proof. Let $\kappa(a)+\kappa(b) > 2\pi$, and so the incident face angles satisfy $\alpha(a) + \alpha(b) < 2\pi$. Let γ be a shortest path on $\mathcal{P}$ connecting a to b. Cut open $\mathcal{P}$ with a cut tree T that includes γ as an edge. How T is completed beyond the endpoints of $\gamma = ab$ doesn't matter.

Let γ_1 and γ_2 be the two sides of the cut γ, and let m_1 and m_2 be the midpoints of γ_1 and γ_2. Reglue the unfolding by folding γ_1 at m_1 and gluing the two halves of γ_1 together, and likewise fold γ_2 at m_2. All the remaining boundary of the unfolding outside of γ is reglued back exactly as it was cut by T.

The midpoint folds at m_1 and m_2 have angle π (because γ is a geodesic). The gluing draws the endpoints a and b together, forming a point with total angle $\alpha(a) + \alpha(b) < 2\pi$. Thus this gluing is an Alexandrov gluing, producing some polyhedron $\mathcal{P}'$. Generically $\mathcal{P}'$ has one more vertex than $\mathcal{P}$: it gains two vertices at m_1 and m_2, and a and b are merged to one. $\mathcal{P}'$ could only have the same number of vertices as $\mathcal{P}$ if $\alpha(a) + \alpha(b) = 2\pi$, which is excluded in this case. □

Lemma 3 *Any polyhedron $\mathcal{P}$ with a pair of vertices each with curvature at most π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*

Proof. Let a and b be a pair of vertices with $\kappa(a) \le \pi$ and $\kappa(b) \le \pi$, and so $\alpha(a) \ge \pi$ and $\alpha(b) \ge \pi$. Let $\gamma = ab$ be a shortest path from a to b on $\mathcal{P}$. Because the curvature at each endpoint is at most π, there is at least π surface angle incident to a and to b. This permits identification of a rectangular neighborhood R on $\mathcal{P}$ with midline ab, whose interior is vertex-free.

Now we select a cut tree T that includes ab and otherwise does not intersect R. This is always possible because there is at least π surface angle incident to both a and b. So we could continue the path beyond ab to avoid cutting into R. Let T unfold $\mathcal{P}$ to polygon P. We will modify T to a new cut tree T'.

Replace ab in T by three edges ab', $b'a'$, $a'b$, forming a zigzag 'Z'-shape, $Z = ab'a'b$, with $Z \subset R$. We will illustrate with an unfolding of a cube, shown in Fig. 1, with ab the edge cut between the front (F) and top (T) faces of the cube.

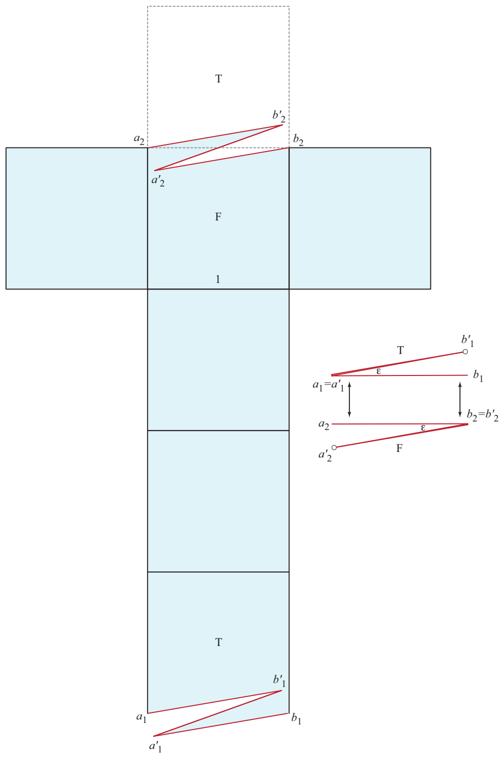

Figure 1: Unfolding of a unit cube. The cut edge ab is replaced by $Z = ab'a'b$. The unfolding P' is illustrated. The insert shows the gluing in the vicinity of ab in the refolding to $\mathcal{P}'$.

We select an angle ε determining the Z according to two criteria. First, ε is smaller than either $\kappa(a)$ and $\kappa(b)$. Second, ε is small enough so that the following construction sits inside R. Let $R' \subset R$ be a rectangle whose diagonal is ab; refer to Fig. 2. Trisect the left

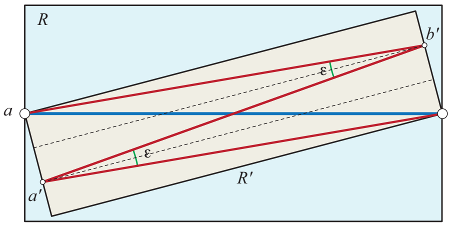

Figure 2: Construction of zig-zag path Z.

and right sides of R', and place a' and b' two-thirds away from a and b respectively. The angle of the Z at a' and at b' is ε. $\triangle ab'a'$ and $\triangle ba'b'$ are congruent isosceles triangles; so $|ab'| = |b'a'| = |a'b|$.

The turn points a' and b' have curvature zero on $\mathcal{P}$ (because $Z \subset R$ and R is vertex-free). Let P' be the polygon obtained by unfolding $\mathcal{P}$ by cutting T', and label the pair of images of each Z corner $a_1, a_1', \ldots, b_2$, as illustrated in Fig. 1. Now we refold it differently, to obtain a different polyhedron $\mathcal{P}'$. "Zip" P' closed at the reflex vertices a_2' and b_1'. Zipping at a_2' glues $a_2'b_2'$ to $a_2'b_2$, so that now $b_2 = b_2'$; zipping at b_1' glues $b_1'a_1'$ to $b_1'a_1$, so that now $a_1 = a_1'$. (See the insert of Fig. 1.) Finally, the two "halves" of the new $a_1'b_1' = a_2b_2$ are glued together, and the remainder of T' is reglued just as it was in T.

This gluing produces new vertices near a' and b', each of curvature $\kappa(a') = \kappa(b') = \varepsilon$. An extra ε of surface angle is glued to both a and b, so their curvatures each decrease by ε (and so maintain the Gauss-Bonnet sum of 4π). By the choice of ε, these curvatures remain positive. Alexandrov's theorem is satisfied everywhere: the curvatures at a, b, a', b' are all positive, and the lengths of the two halves of the new $a_1'b_1' = a_2b_2$ edge are the same (and note this length is not the original length of ab on $\mathcal{P}$, but rather the side-length of the isosceles triangles: $|ab'| = |b'a'| = |a'b|$). So this refolding corresponds to some polyhedron $\mathcal{P}'$. It is different from $\mathcal{P}$ because it has two more vertices at a' and b' (vertices at a and b remain with some positive curvature by our choice of ε). □

Putting Lemmas 2 and 3 together yields the claim:

Theorem 4 *Every polyhedron has an unfolding that refolds to a different polyhedron, i.e., no polyhedron is refold-rigid.*

4 Many (Semi-)Regular Polyhedra are Edge-Unfold Rigid

Our results on edge-unfold rigidity rely on this theorem:

Theorem 5 *Let $\theta_{\min}$ be the smallest angle of any face of $\mathcal{P}$, and let $\kappa_{\max}$ be the largest curvature at any vertex of $\mathcal{P}$. If $\theta_{\min} > \kappa_{\max}$, then $\mathcal{P}$ is edge-unfold rigid.*

Proof. Let T be an edge-unfold cut tree for $\mathcal{P}$, and P the resulting unfolded polygon. No angle on the boundary of P can be smaller than $\theta_{\min}$. Let x be a leaf node of T and y the parent of x. The exterior angle at x in the unfolding P is at most $\kappa_{\max}$. Because every internal angle of P is at least $\theta_{\min}$, which is larger than $\kappa_{\max}$, no point of P can be glued into x, leaving the only option to "zip" together the two cut edges deriving from $xy \in T$. Let $T' = T \setminus xy$ be the cut tree remaining after this partial gluing, and P' the partially reglued manifold.

If T' is not empty, it is a tree, with at least two leaves, one of which might be y (if x was the only child of y). Any leaf $z \in T'$ corresponds to some vertex $v \in \mathcal{P}$, with all but one incident edge already glued. Because P' has not gained any new angles beyond those available in P, we have returned to the same situation: no angle of P' is small enough to fit into the angle gap at z, which is at most $\kappa_{\max}$ at any v. Thus again the edge of T' incident to z must be zipped in the gluing. Continuing in this manner, we see that T may only be reglued by exactly identifying every cut-edge pair, reproducing $\mathcal{P}$. □

Corollary 6 *The regular and semi-regular solids that satisfy Theorem 5, listed in Table 4, are all edge-refold rigid.*

Corollary 7 *Any polyhedron $\mathcal{P}$ that satisfies Theorem 5, may be "perturbed" by moving its vertices slightly to create an uncountable number of edge-refold rigid polyhedra.*

Proof. Proof omitted. □

5 Tetrahedra are edge-unfold transformers

The goal of this section is to prove this theorem:

Theorem 8 *Every tetrahedron may be edge-unfolded and refolded to a different polyhedron.*

There are 16 distinct edge unfoldings of a tetrahedron $\mathcal{T}$. The spanning cut trees that determine these unfoldings fall into just two combinatorial types: The cut tree is a star, a Y-shaped "trident" with three leaves, or the cut tree is a path of three edges. There are 4 different tridents, and $2 \cdot \binom{4}{2} = 12$ different paths. In all these unfoldings, the polygon P that constitutes the unfolded surface is a hexagon: the three cut edges becomes three pairs of equal-length edges of the hexagon. Our goal is to show that, for any $\mathcal{T}$, at least one of the 16 unfoldings P may be refolded to a polyhedron $\mathcal{P}'$ not congruent to $\mathcal{T}$.

Polyhedron Name	$\theta_{\min}$	$\kappa_{\max}$	$\theta_{\min} > \kappa_{\max}$
Dodecahedron	$\frac{3}{5}$	$\frac{1}{5}$	✓
Trunc. Cube	$\frac{1}{3}$	$\frac{1}{6}$	✓
Rhombicuboctahedron	$\frac{1}{3}$	$\frac{1}{6}$	✓
Trunc. Cuboctahedron	$\frac{1}{2}$	$\frac{1}{12}$	✓
Snub Cube	$\frac{1}{3}$	$\frac{1}{6}$	✓
Icosidodecahedron	$\frac{1}{3}$	$\frac{2}{15}$	✓
Trunc. Dodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Trunc. Icosahedron	$\frac{3}{5}$	$\frac{1}{15}$	✓
Rhomb-icosidodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Trunc. Icosidodecahedron	$\frac{1}{2}$	$\frac{1}{30}$	✓
Snub Dodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Pseudo-rhombicuboctahedron	$\frac{1}{3}$	$\frac{1}{6}$	✓

Table 1: Inventory of minimum face angles and maximum vertex curvatures, for selected regular and semi-regular polyhedra. All angles expressed in units of π.

Proof. (of Theorem 8). The proof classifies tetrahedra by their four curvatures, and then establishes the claim for each of the resulting four classes. A concrete example of one of the classes, a *2r-tetrahedron* with $\kappa_1 \geq \kappa_2 \geq 1 > \kappa_3 \geq \kappa_4$, is shown in Fig. 3. Proof omitted. □

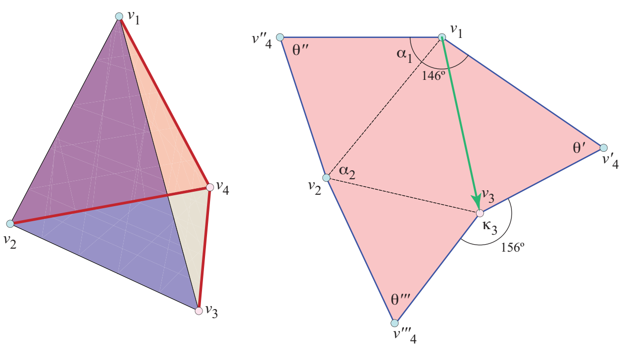

Figure 3: A tetrahedron with v_1, v_2 "convex" and v_3, v_4 "reflex," cut open with a trident rooted at v_4, producing a hexagon with one reflex vertex with exterior angle κ_3. The proof shows that the convex angle α_1 derived from v_1 fits within κ_3.

6 Open Problems

Our work so far just scratches the surface of a potentially rich topic. Here we list some questions suggested by our investigations.

1. Star unfoldings (e.g., [4, Sec. 24.3]) are natural candidates for rigidity. Is it the case that almost every star unfolding of (almost?) every polyhedron is refold-rigid?
2. Which (if either) of the following is true? (a) Almost all polyhedra are edge-unfold rigid. (b) Almost all polyhedra are edge-unfold transformers.
3. Characterize the polygons P that can fold in two different ways (have two different Alexandrov gluings) to produce the exact same polyhedron $\mathcal{P}$. We have only sporadic examples of this phenomenon (among the foldings of the Latin cross).
4. Do our transformer results extend to the situation where the unfoldings are required to avoid overlap? We can extend Lemma 2 to ensure nonoverlap, but extending Lemma 3 seems more difficult.
5. One could view an edge-unfold and refold operation as a directed edge between two polyhedra in the space of all convex polyhedra. Thm. 5 and Cor. 7 show neighbors of some (semi-)regular polyhedra have no outgoing edges. What is the connected component structure of this space?

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