Locked and Unlocked Polygonal Chains in 3D*

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Abstract

In this paper, we study movements of simple polygonal chains in 3D. We say that an open, simple polygonal chain can be *straightened* if it can be continuously reconfigured to a straight sequence of segments in such a manner that both the length of each link and the simplicity of the chain are maintained throughout the movement. The analogous concept for closed chains is *convexification*: reconfiguration to a planar convex polygon. Chains that cannot be straightened or convexified are called *locked*. While there are open chains in 3D that are locked, we show that if an open chain has a simple orthogonal projection onto some plane, it can be straightened. For closed chains, we show that there are unknotted but locked closed chains, and we provide an algorithm for convexifying a planar simple polygon in 3D with a polynomial number of moves.

1 Introduction

A *polygonal chain* $P = (v_0, v_1, \ldots, v_n)$ is a sequence of consecutively joined segments (or edges) $e_i = v_i v_{i+1}$ of fixed lengths $\ell_i = |e_i|$, embedded in space. A chain is *closed* (a *polygon*) if the line segments are joined in cyclic fashion, i.e., if $v_n = v_0$; otherwise, it is *open*. Basic questions concerning reconfiguration of open and closed chains have proved surprisingly difficult. For example, the question of whether every planar, simple open chain can be straightened in the plane while maintaining simplicity has circulated in the computational geometry community for years, but remains open at this writing. Previous computational geometry research on the reconfiguration of chains typically concerns planar chains with crossing links, moving in the presence of obstacles; or reconfigures closed chains with crossing links in dimensions $d \geq 2$ [LW95]. In contrast, throughout this paper we work in 3D and require that chains remain simple throughout their motions. The Schwartz-Sharir cell decomposition approach [SS83] from algorithmic robotics shows that all the problems we consider in this paper are decidable, and Canny's roadmap algorithm [Can87] leads to solutions that are singly exponential in n. Our goal is therefore polynomial-time algorithms.

2 Open Chains with Simple Projections

Our first results are algorithms to straighten open polygonal chains that satisfy either one of two projection conditions. Our algorithms compute reconfigurations that are sequences of "moves." During each move, a (small) constant number of individual joint moves occur, where for each a vertex v_{i+1} rotates monotonically about an axis through joint v_i, with the axis of rotation fixed in a reference frame attached to some edges.

THEOREM 2.1. *If an open polygonal chain of n links either has a simple orthogonal projection onto a plane, or it lies on the surface of a convex polytope, then it may be straightened in $O(n)$ moves. The algorithms run in time polynomial in n.*

3 Locked Chains

We next show that not all open chains may be straightened. Consider the chain $K = (v_0, \ldots, v_5)$ configured as in Fig. 1. One can think of K as composed of two rigid knitting needles, e_0 and e_4, connected by a flexible cord of length $L = \ell_1 + \ell_2 + \ell_3$. By appropriate choice of link lengths and radius r of a ball B centered on v_1, it can be shown that v_0 and v_5 remain exterior to B throughout any motion. This permits completing a trefoil knot exterior to B, which would be unknotted if K were straightened. By contradiction, then, K is locked.

By "doubling" K and joining endpoints, we prove the same result for closed chains. These results were established independently in [CJ99].

THEOREM 3.1. *There exist locked open and locked closed chains.*

4 Convexifying Planar Simple Polygons

A closed chain in a plane, i.e., a planar polygon, may be convexified in 3D by "flipping" out the reflex

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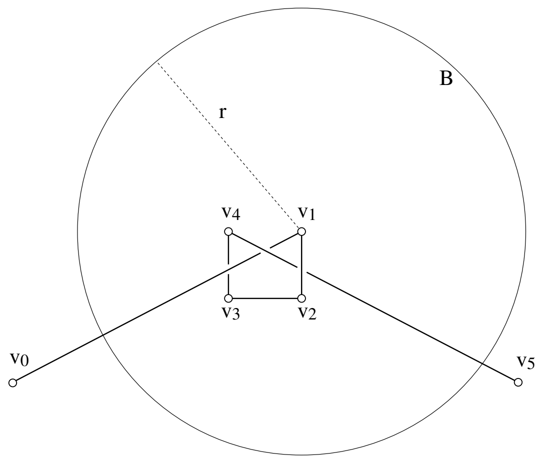

Figure 1: A locked open chain K ("knitting needles").

pockets, i.e., rotating the pocket chain into 3D and back down to the plane. This simple procedure was suggested by Erdős [Erd35] and proved to work by de Sz. Nagy [dSN39]. The number of flips, however, cannot be bound as a function of the number of vertices n of the polygon, as first proved by Joss and Shannon [Grü95].

We offer a new algorithm for convexifying planar closed chains, which we call the "St. Louis Arch" algorithm. It is more complicated than flipping but uses a bounded number of moves. It models the intuitive approach of picking up the polygon into 3D. We discretize this to lifting vertices one by one, accumulating the attached links into a convex "arch" A in a vertical half-plane above the remaining polygonal chain. Although the algorithm is conceptually simple, some care is required to make it precise, and to then establish that simplicity is maintained throughout the motions.

Let P be a simple polygon in the xy-plane, Π_{xy}. Let Π_ε be the plane $z = \varepsilon$ parallel to Π_{xy}, for $\varepsilon > 0$. The value of ε is determined by the initial geometry of P in a complex way. We use this plane to convexify the arch safely above the portion of the polygon not yet picked up. We use primes to indicate positions of moved (raised) vertices. Let $P[i,j]$ represent the chain $(v_i, v_{i+1}, \ldots, v_j)$, including v_i and v_j (where $0 \le i < j < n$), and let $P(i,j)$ represent the chain without its endpoints.

After a generic step i of the algorithm, $P(0,i)$ has been lifted above Π_ε and convexified, v_0 and v_i have been raised to v'_0 and v'_i on Π_ε, and $P[i+1, n-1]$ remains in its original position on Π_{xy}. See Fig. 2.

Next v_{i+1} is lifted to Π_ε, the arch A is rotated down to lie in Π_ε as well, and the resulting "barbed polygon" is convexified within Π_ε. We define a planar polygon as *barbed* if removal of one ear leaves a convex polygon, and prove that every barbed polygon (even "weakly simple"

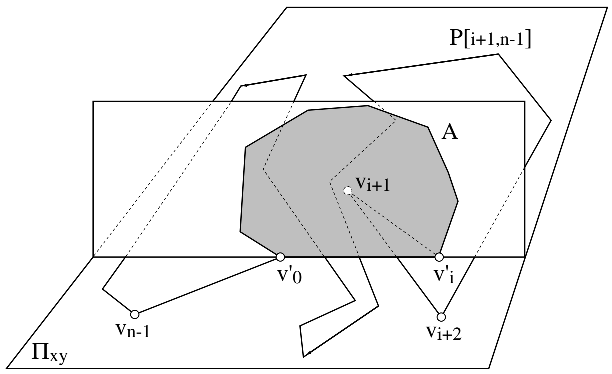

Figure 2: The arch A after the ith step, i.e., after "picking up" $P(0,i)$ into A. (The planes Π_{xy} and Π_ε are not distinguished in this figure.)

ones) may be convexified in its plane in $O(i)$ moves. After convexification, the arch is rotated up into the vertical plane containing the new arch base $v'_0 v'_{i+1}$, and the procedure is repeated.

Theorem 4.1. *The "St. Louis Arch" Algorithm convexifies a planar simple polygon of n vertices in $O(n^2)$ moves; it runs in time polynomial in n.*

5 Open Problems

Two of the most prominent among the many open problems suggested by our work are:

1. What is the complexity of deciding whether a chain (open or closed) in 3D is locked?
2. Can a closed chain with a simple projection always be convexified?

References

[Can87] J. Canny. *The Complexity of Robot Motion Planning.* ACM – MIT Press Doctoral Dissertation Award Series. MIT Press, Cambridge, MA, 1987.

[CJ99] J. Cantarella and H. Johnston. Nontrivial embeddings of polygonal intervals and unknots in 3-space. *J. Knot Theory Ramifications*, 1999. To appear.

[dSN39] B. de Sz. Nagy. Solution to problem 3763. *Amer. Math. Monthly*, 46:176–177, 1939.

[Erd35] P. Erdős. Problem 3763. *Amer. Math. Monthly*, 42:627, 1935.

[Grü95] B. Grünbaum. How to convexify a polygon. *Geombinatorics*, 5:24–30, July 1995.

[LW95] W. J. Lenhart and S. H. Whitesides. Reconfiguring closed polygonal chains in Euclidean d-space. *Discrete Comput. Geom.*, 13:123–140, 1995.

[SS83] J. T. Schwartz and M. Sharir. On the "piano movers" problem II: General techniques for computing topological properties of real algebraic manifolds. *Adv. Appl. Math.*, 4:298–351, 1983.