Sliding-Coin Puzzles

Erik D. Demaine Martin L. Demaine*

In what ways can an arrangement of coins be reconfigured by a sequence of moves where each move slides one coin and places it next to at least two other coins? Martin Gardner publicized this family of sliding-coin puzzles (among others) in 1966. Recently, a general form of such puzzles was solved both mathematically and algorithmically. We describe the known results on this problem, and show several examples in honor of Martin Gardner for the 5th Gathering for Gardner.

1 Puzzles

Sliding-coin puzzles ask you to re-arrange a collection of coins from one configuration to another using the fewest possible moves. Coins are identical in size, but may be distinguished by labels; in some of our examples, the coins are labeled with the letters G, A, R, D, N, E, R. For our purposes, a *move* involves sliding any coin to a new position that touches at least two other coins, without disturbing any other coins during the motion.

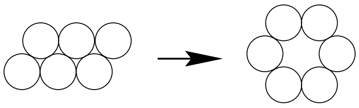
The rest of this section presents several sliding-coin puzzles.

1.1 Triangular Lattice

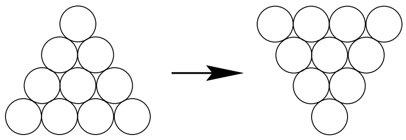
We start with some basic puzzles that are *on the triangular lattice* in the sense that the center of every coin is at a vertex of the planar lattice of equilateral triangles. The restriction that a move must bring a coin to a new position that touches at least two other coins forces a puzzle to stay on the triangular lattice if it is originally on it.

TRIANGULAR LATTICE PUZZLES

Classic Puzzles



Puzzle 1: Rhombus to a circle (3 moves)

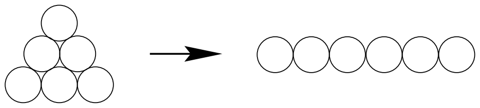


Puzzle 2: Turn the pyramid upside-down (3 moves)

---

*The Demaines are a father-and-son team. Erik D. Demaine, the father, is an assistant professor of computer science at MIT who likes anything algorithmic or geometric. Martin L. Demaine, the son, is a researcher in computer science at MIT who likes anything mathematical and artistic.

New Puzzles



Puzzle 3: Pyramid to a line (7 moves). Source unknown.

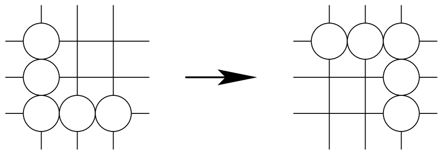
Puzzle 4: Spread out the GARDEN (9 moves).

1.2 Square Lattice

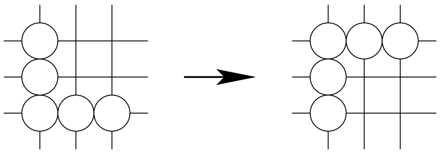
Next we give a few puzzles *on the square lattice*. Here the centers of the coins are at vertices of the planar lattice of squares, and we make the additional constraint that every move brings a coin to such a position. The restriction that a move must bring a coin to a new position that touches at least two other coins does not force the puzzle to stay on the square lattice, but this additional constraint does.

Square Lattice Puzzles

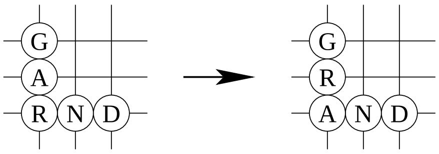
Penta Puzzles



Puzzle 5: Flip the L (4 moves)

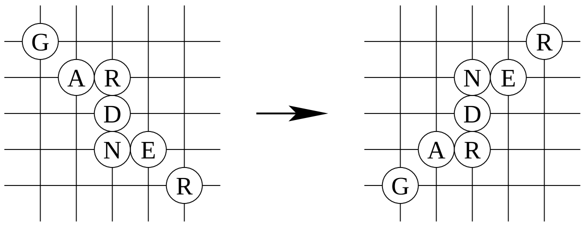


Puzzle 6: Rotate the L by $90°$ (8 moves)

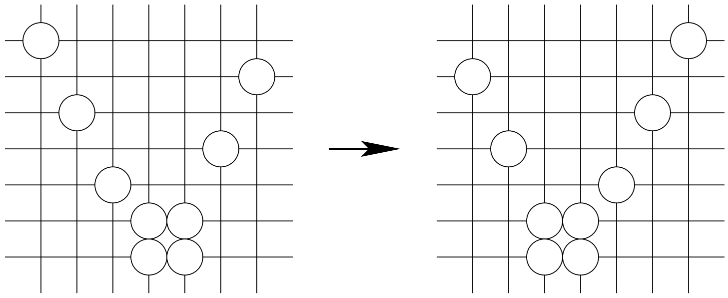


Puzzle 7: Fix the spelling of GRAND (8 moves)

Hard Puzzles



Puzzle 8: Flip the diagonal (hard)



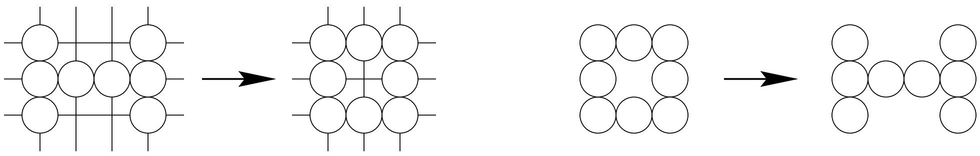
Puzzle 9: Flip the roman numeral V (very hard)

2 History

Sliding-coin puzzles have long been popular. For example, the classic Puzzle 1 is described in Martin Gardner's *Scientific American* article on "Penny Puzzles" [8], in *Winning Ways* [1], in *Tokyo Puzzles* [7], in *Moscow Puzzles* [9], and in *The Penguin Book of Curious and Interesting*

Puzzles [12]. Langman [10] shows all 24 ways to solve this puzzle in three moves. Puzzle 2 is another classic [2, 7, 8, 12]. Other puzzles are presented by Dudeney [6], Fujimura [7], and Brooke [4].

The historical puzzles described so far are all on the triangular lattice. Puzzles on the square lattice appear less often in the literature but have significantly more structure and can be more difficult. The only published example we are aware of is given by Langman [11], which is also described by Brooke [4], Bolt [3], and Wells [12]; see Puzzle 10. However, the second of these puzzles does not remain on the square lattice; it only starts on the square lattice, and the only restriction on moves is that the new position of a coin is adjacent to at least two other coins.



Puzzle 10: Re-arrange the H into the O in four moves while staying on the square lattice (and always moving adjacent to two other coins), and return to the H in six moves using both the triangular and square lattices.

3 Mathematics

A paper by Helena Verrill and the present authors [5] solves a large portion of the general sliding-coin puzzle-solving problem: given two configuration of coins, is it possible to re-arrange the first configuration into the second via a sequence of moves? One catch is that, for the results to apply, a move must be redefined to allow a coin to be picked up and placed instead of just slid on the table. another catch is that the solution does not say anything about the minimum number of moves required to solve a specific puzzle, though it does provide a polynomial upper bound on the number of moves required. (From this information we can also determine which puzzle requires roughly the most moves, among all puzzles.) Despite these catches, the results often apply directly to sliding-coin puzzles and tell us whether a given puzzle is solvable, and if so, how to solve it. The ability to tell whether a puzzle is solvable is ideal for puzzle design.

A surprising aspect of this work is that there is an efficient algorithm to solve most sliding-coin puzzles, which runs fast even for very large puzzles. In contrast, most other games and puzzles, when scaled up sufficiently large, are computationally intractable.

3.1 Triangular Lattice

It turns out to be fairly easy to characterize which triangular-lattice puzzles are solvable. Part of what makes this characterization easy is that most puzzles are solvable. Consider a puzzle with an initial configuration that differs from the goal configuration. There are a few basic restrictions for this puzzle to be solvable:

1. There must be at least one valid move from the initial configuration.
2. The number of coins must be the same for the initial and goal configurations.
3. At least one of the following four conditions must hold:
   (a) The final configuration contains a triangle of three mutually touching coins.
   (b) The final configuration contains four connected coins.
   (c) The final configuration contains three connected coins and two different touching coins (as in Puzzle 4).
   (d) The puzzle is solvable by a single move.
4. If the coins are labeled and there are only three coins (a rather boring situation), then the goal configuration must follow the same 3-coloring of the triangular lattice.

After some thought, you will probably see why each of these conditions must hold for a puzzle to be solvable. What is more surprising, but beyond the scope of this article, is that these conditions are enough to guarantee that the puzzle is solvable. Interested readers are referred to [5] for the proof.

3.2 Square Lattice

Solvable square-lattice puzzles are trickier to characterize. Much more stringent conditions must hold. For example, it is impossible for a configuration of coins to get outside its enclosing box. This property is quite different from the triangular lattice, where coins can travel arbitrary distances.

A notion that turns out to be particularly important with square lattices is the *span* of a configuration. Figure 1 shows an example. Suppose we had a bag full of extra coins, and we could place them onto the lattice at any empty position adjacent to at least two other coins. If we repeat this process for as long as possible, we obtain the span of the configuration.

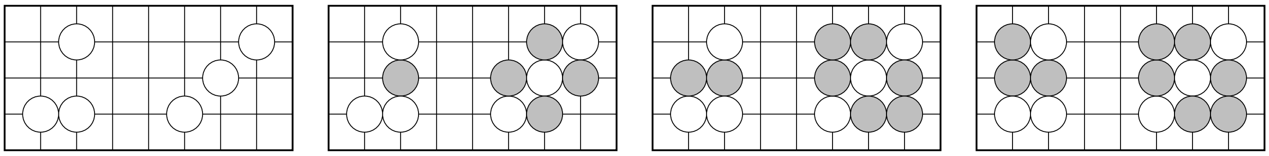


Figure 1: In general, the span consists of one or more rectangles separated by distance at least two empty spaces.

A key property of span is that it can never get larger by a sequence of moves. The span effectively represents all the possibly reachable positions in a configuration. So if you are to move the coins from one configuration to another, the span of the first configuration better contain the span of the second configuration.

This condition is not quite enough, though. In fact, the exact conditions are not known for when a square-lattice puzzle can be solved. However, most solvable configurations have some *extra coins* whose removal would not change the span of the configuration. The main result of [5] says that if a configuration has at least two extra coins, then it can reach any other configuration with the same or smaller span.

This result is complicated, leading to some puzzles with intricate solutions, as in Puzzles 8 and 9, for example. To work your way up to these difficult puzzles, we have provided some "warm up" puzzles that involve 5 coins, in the spirit of the "penta" theme of the Fifth Gathering for Gardner.

References

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. A solitaire-like puzzle and some coin-sliding problems. In *Winning Ways*, volume 2, pages 755–756. Academic Press, London, 1982.

[2] Brian Bolt. Invert the triangle. In *The Amazing Mathematical Amusement Arcade*, amusement 53, page 30. Cambridge University Press, Cambridge, 1984.

[3] Brian Bolt. A two touching transformation. In *Mathematical Cavalcade*, puzzle 20, page 10. Cambridge University Press, Cambridge, 1991.

[4] Maxey Brooke. *Fun for the Money*. Charles Scriber's Sons, New York, 1963. Reprinted as *Coin Games and Puzzles* by Dover Publications, 1973.

[5] Erik D. Demaine, Martin L. Demaine, and Helena Verrill. Coin-moving puzzles. In R. J. Nowakowski, editor, *More Games of No Chance*, pages 405–431. Cambridge University Press, 2002. Collection of papers from the MSRI Combinatorial Game Theory Research Workshop, Berkeley, California, July 24–28, 2000. `http://www.arXiv.org/abs/cs.DM/0204002`.

[6] Henry Ernest Dudeney. "The four pennies" and "The six pennies". In *536 Puzzles & Curious Problems*, problems 382–383, page 138. Charles Scribner's Sons, New York, 1967.

[7] Kobon Fujimura. "Coin pyramids," "Four pennies," "Six pennies," and "Five coins". In *The Tokyo Puzzles*, puzzles 23 and 25–27, pages 29–33. Charles Scribner's Sons, New York, 1978.

[8] Martin Gardner. Penny puzzles. In *Mathematical Carnival*, chapter 2, pages 12–26. Alfred A. Knopf, New York, 1975. Appeared in *Scientific American*, 214(2):112–118, February 1966, with solutions in 214(3):116–117, March 1966.

[9] Boris A. Kordemsky. A ring of disks. In *The Moscow Puzzles*, problem 117, page 47. Charles Scribner's Sons, New York, 1972.

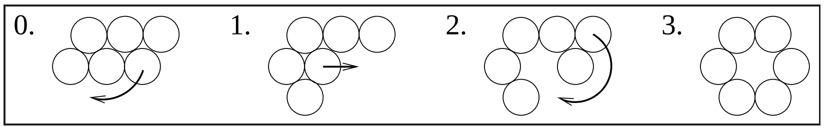
[10] Harry Langman. Curiosa 261: A disc puzzle. *Scripta Mathematica*, 17(1–2):144, March–June 1951.

[11] Harry Langman. Curiosa 342: Easy but not obvious. *Scripta Mathematica*, 19(4):242, December 1953.

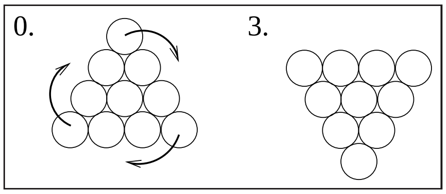
[12] David Wells. "Six pennies," "OH-HO," and "Inverted triangle". In *The Penguin Book of Curious and Interesting Puzzles*, puzzle 305, 375, and 376, pages 101–102 and 125. Penguin Books, 1992.

A Solutions

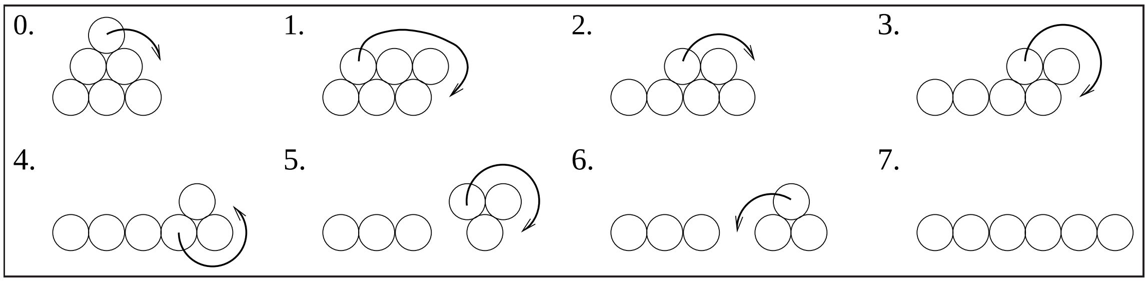
Solutions to Puzzles 1, 2, 3, 5, 6, and 7 were found by an exhaustive breadth-first search, and as a result we are sure that the solutions use the fewest possible moves. For Puzzle 4, it is conceivable that using more than two "rows" leads to a shorter solution; the solution below is the best among all two-row solutions. Puzzles 8 and 9 are left as challenges to the reader; we do not know the minimum number of moves required to solve them. The second half of Puzzle 10 was solved by hand, but the number of moves is minimum: the maximum overlap between the H and the O is four coins, three moves are necessary to enter the triangular lattice and return to the square lattice, and every move starting with Move 3 puts a coin in its final position.



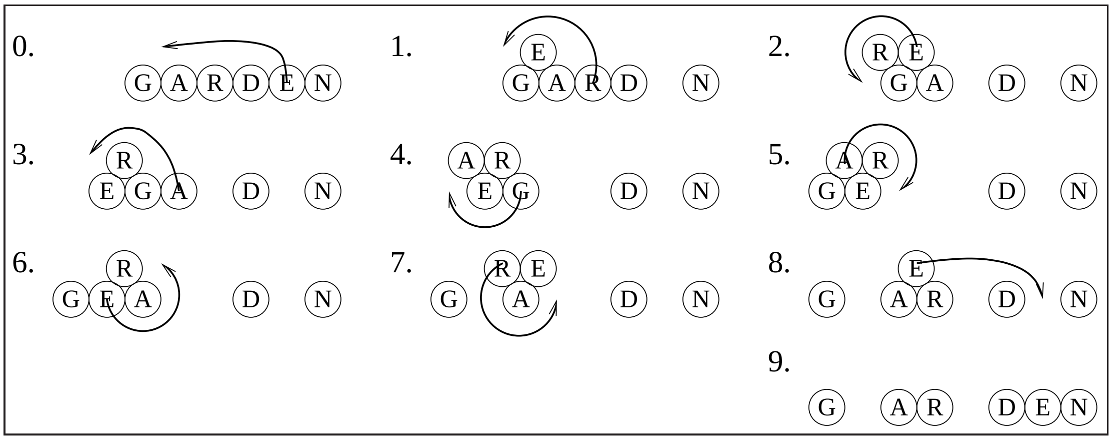
Solution to Puzzle 1: Three moves.



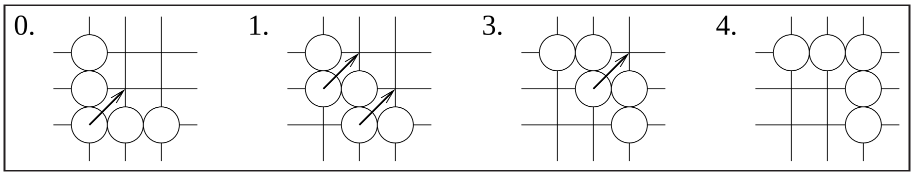
Solution to Puzzle 2: Three moves.



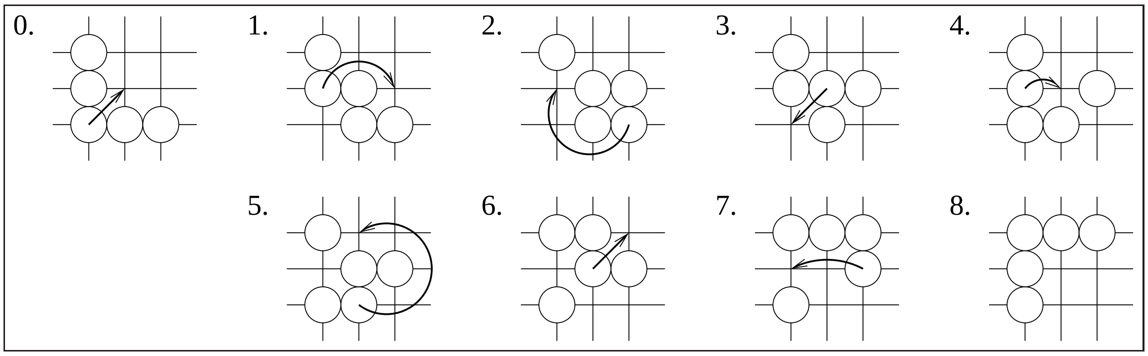
Solution to Puzzle 3: Seven moves.



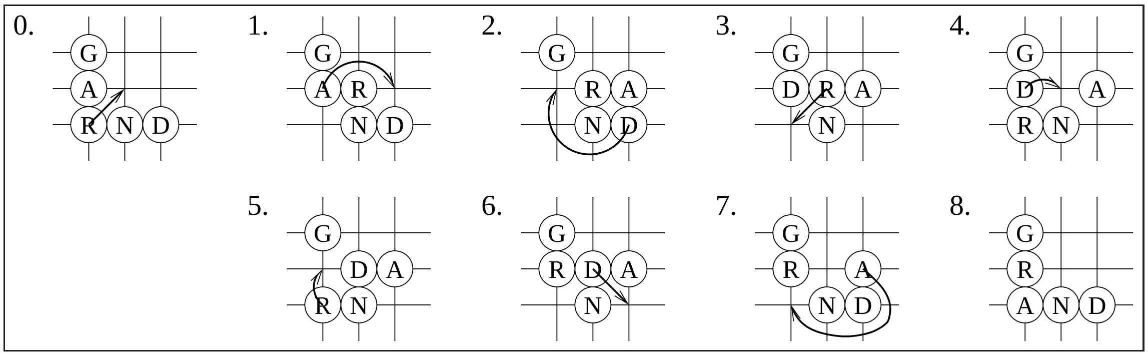
Solution to Puzzle 4: Nine moves.



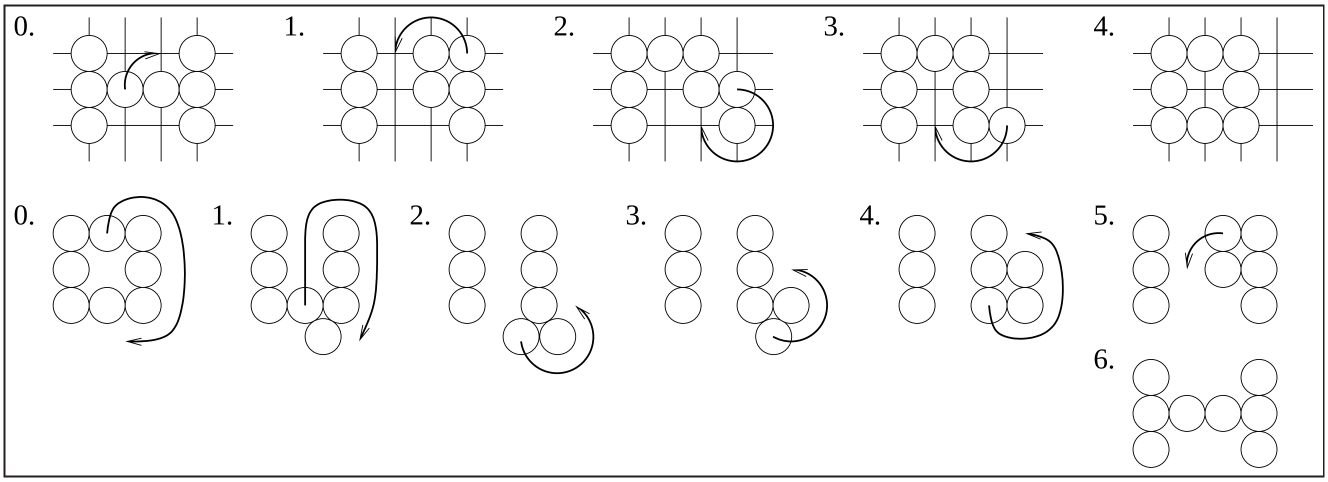
Solution to Puzzle 5: Four moves.



Solution to Puzzle 6: Eight moves.



Solution to Puzzle 7: Eight moves.



Solution to Puzzle 10: Four moves there and six moves back.