Curves in the Sand: Algorithmic Drawing

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1 Introduction

Ethnomathematics is the study of mathematics in the works of art of various cultures [3, 4, 10, 14]. The concepts in this paper are inspired by the visual art of *sand drawings* that has developed independently in different forms in diverse cultures. Generally speaking, the artist draws a set of dots on some flat surface (usually in the sand or in powder on the floor) and then draws one continuous curve that surrounds the dots and crosses itself repeatedly. Although not universally the case, we focus on drawings in which there is exactly one dot per bounded face (and no dots in the outside face). In particular, sand drawings made by the *Tshokwe* people in the West Central Bantu area of Africa are called *sona*.

Sona drawings have been considered in the field of topology under an equivalent guise as generic planar closed curves (immersions of the unit circle into the plane). Several topological invariants about such curves are proved by Arnol′d [2], who also enumerated all sona drawings on small numbers of dots. Carvalho [5] considers curves that are "maximally looped". Ozawa [12] considers the number of bitangents, tangents shared by different points on the curve.

From a graph-theoretic perspective, sona drawings can be viewed as 4-regular planar maps with the additional property that some Eulerian cycle "goes straight" at every vertex. This class of underlying graphs is called *Gaussian graphs*; the name is attributed to an observation made by Carl Gauss in 1830 [9] that was proved by Julius v. Sz. Nagy almost a hundred years later [13]. More recently, Michael Gargano and John Kennedy [8] introduced a more formal notion of Gaussian graphs, which was later generalized by John Kennedy and Brigitte and Herman Servatius [11]. The connection between Gaussian graphs, generic closed curves, and sona drawings was first unveiled in [6], where many open problems about the topic are also posed.

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In this paper we describe algorithms that generate sona drawings under a variety of different models and constraints, in particular settling some of the open questions from [6] and raising several new questions. In particular, we study sona drawings that turn only clockwise and adhere to a given 2-coloring of the points (Section 3), sona drawings that turn only clockwise and minimize the total turn angle (Section 4), polygonal sona drawings with the fewest links (Section 5), and sona drawings on the square grid (Section 6). We also show that the minimum-length sona drawing of a given point set is within a constant factor of a TSP tour (Section 7).

2 Definitions

A *sona drawing* or *sona map* is a closed curve drawn in the plane such that the curve does not touch itself without crossing itself, and no more than two pieces of the curve intersect at the same point. A *sona drawing of a point set* must additionally have exactly one point in each bounded face, and zero points in the outside face. A *sona vertex* is a point at which the curve self-intersects. A *sona edge* is a piece of a curve incident to exactly two sona vertices at its endpoints. A *sona face* is an empty region bounded by a cycle of sona edges. Two sona faces are *adjacent* if they share one or more sona edges. A curve or sona drawing is *clockwise-turning* if it can be drawn continuously with all changes in direction being locally right turns.

3 Two-Color Clockwise-Turning Sona Drawings

In this section, we consider the problem of finding a clockwise-turning sona drawing for a 2-colored point set such that no two adjacent faces contain points of the same color. Because every sona drawing has even vertex degrees, its dual is bipartite, so it has a face 2-coloring; the goal is to make this 2-coloring of the faces match the given coloring of the corresponding points inside the faces. As observed in [6, Lemma 13], every 2-colored point set has such a *color-respecting* sona drawing. On the other hand, it is easy to see that every point set has a clockwise-turning sona drawing.

The challenge we address here is to simultaneously satisfy both constraints, color respecting and clockwise turning. This goal is impossible, e.g., for monochromatic point sets in convex position. However, we show that all nonmonochromatic colorings admit a sona drawing of the desired type:

Theorem 1 *Every set of red and blue points in the plane with at least one point of each color admits a color-respecting clockwise-turning sona drawing.*

Proof sketch: First we show that it suffices to consider points that lie arbitrarily close to the x axis, by a generic rotation and scaling of y by ε. Then we show how to decompose the point set S into intervals $S_1, S_2, \ldots, S_k$, the last of which wraps around from right to left, such that each interval consists of zero or more points of one color followed by one *cap* point of the opposite color, and such that the cap colors alternate cyclicly red/blue. Then we visit each interval in order as shown in Figure 1. □

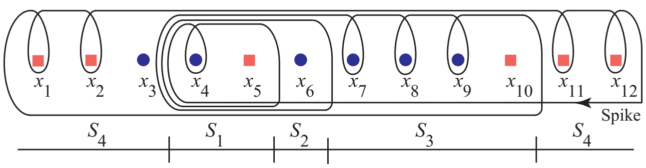


Figure 1: A color-respecting clockwise-turning sona drawing for S. The drawing incrementally incorporates S_1, S_2, S_3, S_4.

Any monochromatic point set with at least one point interior to the convex hull has a color-respecting clockwise-turning sona drawing—a "star" of loops around that point—completing the characterization of point sets with color-respecting clockwise-turning sona drawings.

4 Min-Winding Clockwise-Turning Sona Drawings

In this section, we consider the problem of finding a clockwise-turning sona drawing for a given point set S that has the minimum possible winding number, or equivalently, the minimum possible absolute total turn angle. The *winding number* of a sona drawing is the winding number of the underlying closed curve, that is, the number of complete clockwise turns made by a normal to the curve as we continuously move its base along one complete cycle of the curve. Here we suppose that the curve has finite length and, for simplicity, is differentiable everywhere. Equivalently, the total clockwise turn angle is 360° times the winding number. For clockwise-turning sona drawings, the winding number is always positive, and it can be computed as the number of times a particular normal direction (say, $+x$) occurs as we trace one complete cycle of the curve. Without the clockwise-turning constraint, this problem is similar to the NP-complete angular-metric traveling salesman problem, where the goal is to find a tour of a given set of points with minimum total absolute turn angle [1].

Proposition 2 *The winding number of any clockwise-turning sona drawing of two or more points is at least* 2.

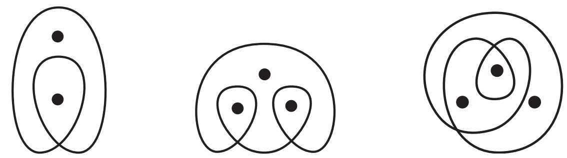
4.1 Points in Convex Position

Proposition 3 *For any set S of n points in convex position, there is a clockwise-turning sona drawing with winding number equal to* 2 *if $|S|$ is even, or* 3 *if $|S|$ is odd.*

Proof sketch: Figure 2 shows the cases $n \in \{2, 3\}$. Figure 3 shows the construction for $n \geq 4$. □

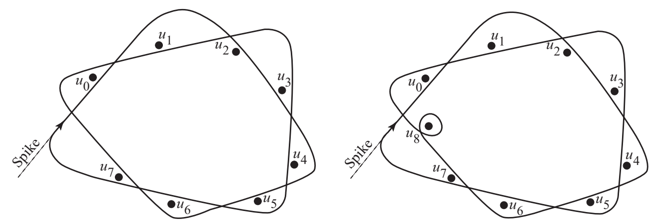
4.2 Convex Peeling Layers

In this section, we relate the minimum winding number to the number of convex-hull (onion) peeling layers:



(a) (b) (c)

Figure 2: Clockwise-turning sona drawings. (a) Winding number is 2. (b–c) Winding number is 3.



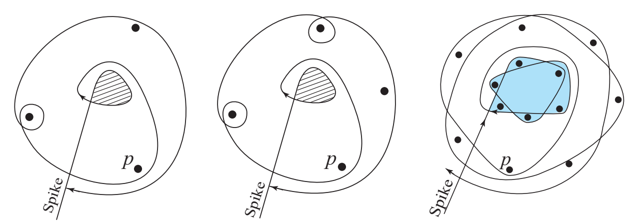
(a) Even number of points, winding number 2.

(b) Odd number of points, winding number 3.

Figure 3: Clockwise-turning sona drawings for convex point sets.

Proposition 4 *For any set S of points in general position decomposing into k nested convex layers, there is a clockwise-turning sona drawing with winding number at most* $4k - 1$.

Proof sketch: We pierce all convex layers with a spike, then visit each layer from the innermost out. In between two layers, we loop around a chosen point p to ensure enough angular freedom to visit the next layer. Figure 4 shows examples of the construction. □



(a) A three-point convex layer.

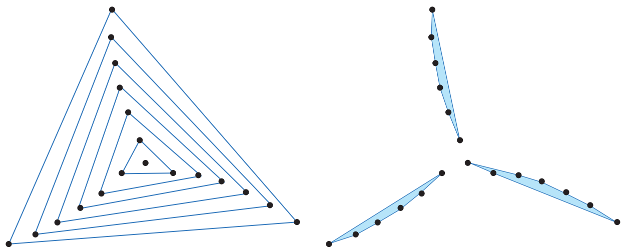
(b) A four-point convex layer.

(c) More than four points.

Figure 4: Transitioning from one layer to the next begins by wrapping around p.

4.3 Radial Convex Partitions

Figure 5 shows a limitation to the convex-layers approach of Section 4.2: the number of convex layers can be large (here, $n/3$) yet the point set can be partitioned into few disjoint convex polygons (here, 3). While it remains open whether the winding number of a clockwise-turning sona drawing is at most a constant factor times the size of such a minimum convex partition, we consider here one type of convex partition which includes the one in Figure 5(b). Namely, a convex partition is *radial* if every convex polygon shares a point with the convex hull of the entire point set.



(a) Nested convex partition. (b) Radial convex partition.

Figure 5: Nested convex partitions are bad approximations to minimum convex partitions.

Proposition 5 *Given a radial convex partition $S_1, S_2, \ldots, S_k$ of a set S of points in general position, we can construct a clockwise-turning sona drawing for S with winding number at most $3k+1$.*

Figure 6 shows an example of our approach applied to a point set similar to Figure 5. For clarity, we have reduced the size of each component in to just three points, but by using the construction from Figure 3, the approach extends to point sets of arbitrary sizes. Note that it was necessary to remove one point from a convex component to cover the middle face.

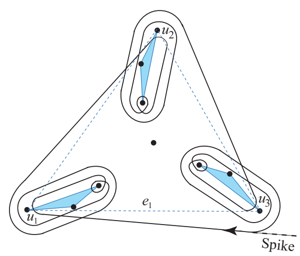


Figure 6: Clockwise-turning sona drawing for a radial convex partition.

5 Minimum-Link Sona Drawings

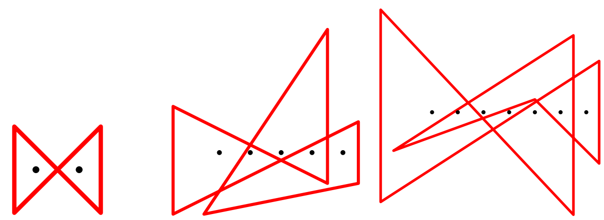
In a *straight-line* sona drawing, the curve is a polygonal chain of line segments called *links*. In this section, we consider the problem of constructing straight-line sona drawings on a given point set with the minimum possible number of links: the *sona link number* $L(S)$ is the minimum number of links of a straight-line sona drawing on the planar point set S. We consider worst-case bounds on this number: $L(n)$ is the maximum $L_S(n)$ over all point sets S of size n. We prove nearly matching upper and lower bounds on $L(n)$.

Straight-line sona drawing closely relates to shattering: separating a given set of objects such that each falls in a single cell of the subdivision of space. In particular, straight-line sona is related to shattering a set of n points in the plane by an arrangements of lines. It is known that $\Omega(\sqrt{n})$ lines are required to shatter n points in general position, and that the problem of finding the minimum number of lines shattering n points is NP-complete [7]. The difference between the problems is that, in sona, we use polygonal chains instead of lines and we do not allow empty regions. However, if we extend the edges of a sona drawing on a set of points, we obtain a shattering of those points. Thus, the shattering number of n points is a lower bound on the minimum number of links in a straight-line sona drawing.

By the scaling transformation described in Section 3, we can assume that the points lie arbitrarily close to the x axis. Call a straight-line sona drawing *boxed* if the leftmost and rightmost links are vertical, the leftmost link is arbitrarily close to the leftmost point, the rightmost link is arbitrarily close to the rightmost point, both of these links are symmetric about the x axis, and all other links are within the vertical strip between these two links.

Lemma 6 *Given n points where $n \in \{2, 5, 7\}$, there is a boxed straight-line sona drawings with four links for $n = 2$, seven links for $n = 5$, and eight links for $n = 7$.*

Proof sketch: See Figure 7. □



(a) 2 points, 4 links. (b) 5 points, 7 links. (c) 7 points, 8 links.

Figure 7: Examples of boxed straight-line sona.

Lemma 7 *Given constructions for boxed sona drawings on n_1 points using e_1 links and on n_2 points using e_2 links, we can construct a boxed sona drawing on $n_1 + n_2 - 1$ given points with $e_1 + e_2 - 2$ links.*

Lemma 8 *For every positive integer n, there is a set of n points requiring $n+1$ links if n is odd, and requiring $n+2$ links if n is even, in any straight-line sona drawing.*

Proof sketch: Consider n collinear points. □

Combining Lemmas 6, 7, and 8, we obtain the main result:

Theorem 9 *Given n points in the plane, the following number of links are sufficient and sometimes necessary for a straight-line sona drawing:*

$$\begin{array}{rcccll}
 & & L(n) & = & 4 & \textit{for } n = 2, \\
4 & \le & L(n) & \le & 6 & \textit{for } n = 3, \\
6 & \le & L(n) & \le & 8 & \textit{for } n = 4, \\
 & & L(n) & = & n+2 & \textit{for } n \equiv 0 \pmod 4 \textit{ with } n \ge 8, \\
n+1 & \le & L(n) & \le & n+2 & \textit{for } n \equiv 1 \pmod 4 \textit{ with } n \ge 1, \\
n+2 & \le & L(n) & \le & n+3 & \textit{for } n \equiv 2 \pmod 4 \textit{ with } n \ge 6, \\
 & & L(n) & = & n+1 & \textit{for } n \equiv 3 \pmod 4 \textit{ with } n \ge 7.
\end{array}$$

6 Grid Sona Drawings

Given n points in the centers of cells in the square grid, a *grid sona drawing* is a sona drawing whose edges are drawn as polygonal lines along the orthogonal grid lines. Not all point sets have a grid sona drawing. However, if the points are "far enough" from each other, then we can always find a grid sona drawing: loop around each point except the last one, surround the last point, and return to the starting position. To quantify "far enough", consider the following process: start with an arbitrary set of points in the centers of grid cells, and then scale the point set by an integer s, so that between any two distinct points there are at least s horizontal or vertical grid lines. Our goal is to find grid sona drawings with the minimum possible scaling s. We prove that $s = 3$ always suffices, and show some instances where $s = 2$ also suffices.

Proposition 10 *Any point set has a (clockwise-turning) grid sona drawing after scaling by* 3.

Proposition 11 *For any positive integers m and n, the $m \times n$ grid of points has a grid sona drawing after scaling by* 2.

Proof sketch: The construction varies slightly depending on the parity of the number of rows and columns. All cases share a common "core" construction, shown in Figure 8. □

7 Minimum-Length Sona Drawings

The *length* of a sona drawing is the total arc length of the underlying curve. In this section, we show that the length of the minimum-length sona drawing is at least a constant factor times the TSP tour of the given points, thus settling the open problem posed in [6, Open Problem 6]. By a matching upper bound of [6], the two values are thus within constant factors of each other.

Theorem 12 *Every sona drawing has length greater than* TSP$/c$ *where* $c = \frac{\pi+2}{\pi} \approx 1.63662$.

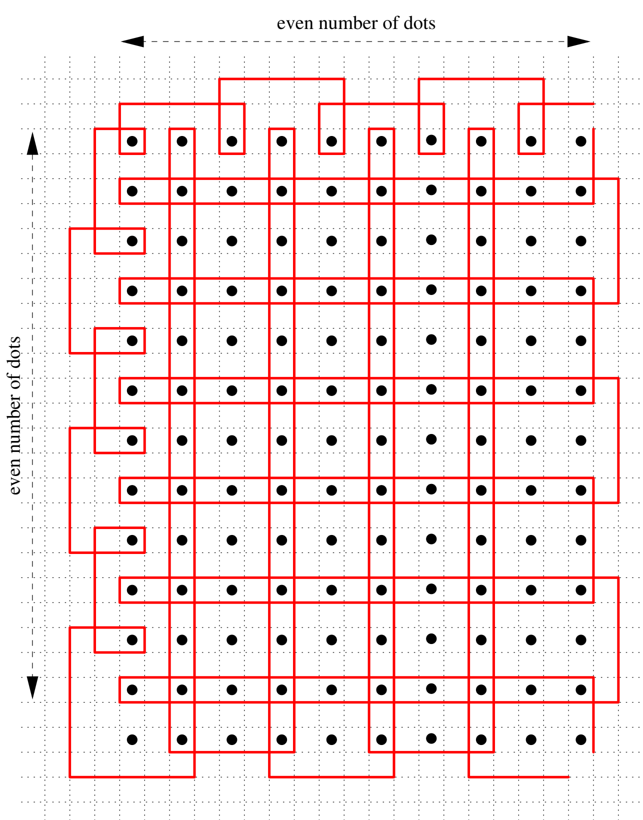


Figure 8: Core construction for grid sona drawing on an $m \times n$ grid of points after scaling by 2. Can you fill in the rest? What if we add one more row and/or column of points?

Proof sketch: We use connections between TSP, the nearest-neighbor graph, and Eulerian tours. □

8 Open Problems

We highlight some open problems:

Open Problem 1 *What is the complexity of finding the minimum-winding clockwise-turning sona drawing on a given set of points?*

Open Problem 2 *Are the upper or lower bounds in Theorem 9 on the minimum-link sona drawing tight?*

Open Problem 3 *Which point sets have grid sona drawings without any scaling? Do all point sets have grid sona drawings after scaling by 2?*

Open Problem 4 *What is the complexity of finding the minimum-length sona drawing on a given set of points?*

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