

# Approximability of the Subset Sum Reconfiguration Problem

Takehiro Ito<sup>1,\*</sup> and Erik D. Demaine<sup>2</sup>

<sup>1</sup> Graduate School of Information Sciences, Tohoku University,  
Aoba-yama 6-6-05, Sendai, 980-8579, Japan

takehiro@ecei.tohoku.ac.jp

<sup>2</sup> MIT Computer Science and Artificial Intelligence Laboratory,  
32 Vassar St., Cambridge, MA 02139, USA

edemaine@mit.edu

**Abstract.** The SUBSET SUM problem is a well-known NP-complete problem in which we wish to find a packing (subset) of items (integers) into a knapsack with capacity  $c$  so that the sum of the integers in the packing is at most the capacity of the knapsack and at least a given integer threshold. In this paper, we study the problem of reconfiguring one packing into another packing by moving only one item at a time, while at all times maintaining the feasibility of packings. First we show that this decision problem is strongly NP-hard, and is PSPACE-complete if we are given a conflict graph for the set of items in which each vertex corresponds to an item and each edge represents a pair of items that are not allowed to be packed together into the knapsack. We then study an optimization version of the problem: we wish to maximize the minimum sum among all packings in the reconfiguration. We show that this maximization problem admits a polynomial-time approximation scheme (PTAS), while the problem is APX-hard if we are given a conflict graph.

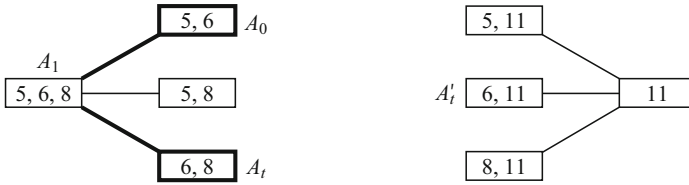
## 1 Introduction

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible. Recently, Ito *et al.* [7] proposed a framework of reconfiguration problems, and gave complexity and approximability results for reconfiguration problems derived from several well-known problems, such as INDEPENDENT SET, CLIQUE, MATCHING, etc. In this paper, we study two reconfiguration problems derived from the SUBSET SUM problem.

The SUBSET SUM problem is a well-known NP-complete problem, defined as follows [9]. Suppose that we are given a knapsack with a nonnegative integer capacity  $c$ , and a set  $A$  of items  $a_1, a_2, \dots, a_n$ , each of which has a nonnegative integer size  $s(a_i)$ ,  $1 \leq i \leq n$ . We call a subset  $A'$  of  $A$  a *packing* if the total size of  $A'$  does not exceed the capacity  $c$ , that is,  $\sum_{a \in A'} s(a) \leq c$ . Given an integer

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**Fig. 1.** All packings of total size at least  $k = 10$  for  $A = \{5, 6, 8, 11\}$  and  $c = 20$

threshold  $k$ , the SUBSET SUM problem is to find a packing  $A'$  whose total size is at least  $k$ , that is,  $k \leq \sum_{a \in A'} s(a) \leq c$ . For a knapsack with capacity  $c = 20$  and a set  $A = \{5, 6, 8, 11\}$ , there are 8 packings of total size at least  $k = 10$ , as illustrated in Fig.1, where each packing is surrounded by a box. Our definition of SUBSET SUM is known as the decision version of the MAXIMUM SUBSET SUM problem in which we wish to find a packing whose total size is maximum [9]<sup>1</sup>.

Suppose now that we are given *two* packings  $A_0$  and  $A_t$ , both of total size at least  $k$ , and we are asked whether we can transform one into the other via packings by moving (namely, either adding or subtracting) a single item to/from the previous one without ever going through a packing of total size less than  $k$ . We call this decision problem the SUBSET SUM RECONFIGURATION problem. For two packings  $A_0 = \{5, 6\}$  and  $A_t = \{6, 8\}$  in Fig.1, the answer is “yes” since they can be transformed into each other via  $A_1 = \{5, 6, 8\}$ ; in Fig.1, two packings (boxes) are joined by a line if and only if one packing can be obtained from the other by moving a single item.

Obviously, we cannot always find such a transformation. For example, there is no transformation between  $A_0 = \{5, 6\}$  and  $A'_t = \{6, 11\}$  in Fig.1 if we are allowed to use only packings of total size at least  $k = 10$ . On the other hand, the answer is always “yes” if  $k = 0$ : we first remove all items of  $A_0$ , and obtain the empty packing; and then, add all items of  $A_t$  to the knapsack. In turn, we can get a natural optimization problem if we wish to maximize the minimum total size among all packings in a transformation between  $A_0$  and  $A_t$ . We call this maximization problem the MAXMIN SUBSET SUM RECONFIGURATION problem. The sequence of packings emphasized by thick lines in Fig.2 is an optimal solution for  $A_0 = \{5, 6\}$  and  $A'_t = \{6, 11\}$ ; its objective value is 8.

Reconfiguration problems have been studied extensively in recent literature [2,5,6,7,8], but reconfiguration problems for SUBSET SUM have not been studied yet. One can easily imagine a variety of practical scenarios, where a packing (e.g., representing a feasible display of electronic advertisements on a Web browser) needs to be changed (to show other advertisements) by individual changes (appealing to the user by showing one by one) while maintaining both threshold and capacity of the allowed area on the Web browser (in order to maintain both advertiser and user satisfactions during the transformation). Reconfiguration problems are also interesting in general because they provide a new perspective

<sup>1</sup> Note that SUBSET SUM in [4] is slightly different from our definition: SUBSET SUM in [4] is defined as the problem of finding a packing whose total size is exactly  $k$ .

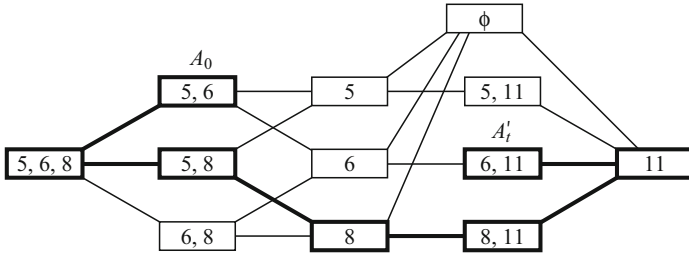


Fig. 2. All packings for  $A = \{5, 6, 8, 11\}$  and  $c = 20$

and deeper understanding of the solution space and of heuristics that navigate that space.

For the (ordinary) SUBSET SUM and MAXIMUM SUBSET SUM problems, several variants have been studied [9]. In particular, MAXIMUM SUBSET SUM with “conflict graph” [10] is an important variant, because this variant has been studied for several other problems, such as BIN PACKING [3] and SCHEDULING under makespan minimization [1]. In the variant, we are given a *conflict graph* for a set  $A$  of items in which each vertex corresponds to an item in  $A$ , and each edge represents a pair of items in  $A$  that are not allowed to be packed together into the knapsack. It is known that the (ordinary) MAXIMUM SUBSET SUM problem with conflict graph is strongly NP-hard [10].

In this paper, we first show that SUBSET SUM RECONFIGURATION is strongly NP-hard, and is PSPACE-complete for the variant of conflict graph. We then show that MAXMIN SUBSET SUM RECONFIGURATION with conflict graph is APX-hard, and hence there is no polynomial-time approximation scheme (PTAS) for this variant unless  $P = NP$ . In contrast, we give a PTAS for the original version of MAXMIN SUBSET SUM RECONFIGURATION. Note that, since this maximization problem is strongly NP-hard, the problem does not admit a fully polynomial-time approximation scheme (FPTAS) unless  $P = NP$ ; in this sense, a PTAS is the best approximation algorithm we can expect for the problem [11, p. 72].

Our main result of this paper is a PTAS for MAXMIN SUBSET SUM RECONFIGURATION. The strategy of our PTAS is the following: we divide a set  $A$  of items into two groups, one is the set of items having “large” sizes, and the other consists of items having “small” sizes; and we deal with the two groups separately. Because such an approximation technique is fairly standard, especially for MAXIMUM SUBSET SUM and BIN PACKING [9,11], one might think that our PTAS could be obtained also straightforwardly by extending several known FPTAS or PTAS [9,11]. However, this is not the case, because the focus of reconfiguration problems is different from the ordinary problems: we seek the *reachability* between two feasible solutions, and hence the placement of items is the central matter. For example, two packings  $\{5, 6\}$  and  $\{11\}$  in Fig.1 have the same total size 11, and hence we can regard them as an “equivalent” packing in the ordinary SUBSET SUM problem. However, we cannot identify these two packings in the reconfiguration problems; for example,  $\{11\}$  can be transformed into  $\{6, 11\}$ ,

but  $\{5, 6\}$  cannot, when  $k = 10$ . (See Fig.1.) We thus introduce a “configuration graph” which represents the placements of items and their connectivity. (A formal definition will be given in Section 3, but an example is already shown in Fig.2.) Our main idea is to approximate the configuration graph appropriately.

## 2 Complexity and Inapproximability

Before showing our results, we introduce some terms and define the problems more formally. In the Introduction, we have defined a packing  $A_i$  as a subset of items in a set  $A$  such that the total size of  $A_i$  is at most the capacity  $c$  of a knapsack; the total size of a packing  $A_i$  is denoted by  $s(A_i)$ , that is,  $s(A_i) = \sum_{a \in A_i} s(a)$ . Note that a packing does not necessarily satisfy a threshold  $k$ . We say that two packings  $A_i$  and  $A_j$  of  $A$  are *adjacent* if their symmetric difference is of cardinality 1, that is,  $|A_i \Delta A_j| = |(A_i \setminus A_j) \cup (A_j \setminus A_i)| = 1$ ; the item  $a$  in  $A_i \Delta A_j$  is said to be *moved* between  $A_i$  and  $A_j$ . A *reconfiguration sequence* between two packings  $A_0$  and  $A_t$  is a sequence of packings  $A_0, A_1, \dots, A_t$  such that  $A_{i-1}$  and  $A_i$  are adjacent for  $i = 1, 2, \dots, t$ . For a reconfiguration sequence  $\mathcal{P}$ , we denote by  $f(\mathcal{P})$  the minimum total size among all packings in  $\mathcal{P}$ , that is,  $f(\mathcal{P}) = \min\{s(A_i) : A_i \in \mathcal{P}\}$ . Then, for two packings  $A_0$  and  $A_t$ , let

$$\text{OPT}(A_0, A_t) = \max\{f(\mathcal{P}) \mid \mathcal{P} \text{ is a reconfiguration sequence between } A_0 \text{ and } A_t\}.$$

Given an integer threshold  $k$  and two packings  $A_0$  and  $A_t$  with  $s(A_0) \geq k$  and  $s(A_t) \geq k$ , the SUBSET SUM RECONFIGURATION problem is a decision problem to determine whether  $\text{OPT}(A_0, A_t) \geq k$ . On the other hand, its optimization version is defined as follows: Given two packings  $A_0$  and  $A_t$ , the MAXMIN SUBSET SUM RECONFIGURATION problem is to compute  $\text{OPT}(A_0, A_t)$ . Note that we are asked simply to compute the optimal value  $\text{OPT}(A_0, A_t)$ , and we need *not* to find an actual reconfiguration sequence.

We first have the following theorem, whose proof is omitted from this extended abstract.

**Theorem 1.** *Both SUBSET SUM RECONFIGURATION and MAXMIN SUBSET SUM RECONFIGURATION are strongly NP-hard.*

We then consider the variant with conflict graph. Notice that every feasible packing of  $A$  induces an independent set of the conflict graph. Therefore, we have the following theorem.

**Theorem 2.** *SUBSET SUM RECONFIGURATION with conflict graph is PSPACE-complete.*

*Proof.* It is easy to see that the problem is in PSPACE. Therefore, we show that SUBSET SUM RECONFIGURATION with conflict graph is PSPACE-hard by giving a polynomial-time reduction from the INDEPENDENT SET RECONFIGURATION problem [7].

Given a graph  $G$  of  $n$  nodes, an integer threshold  $k'$ , and two independent sets  $I_0$  and  $I_t$  of  $G$ , both of cardinality at least  $k'$ , the INDEPENDENT SET RECONFIGURATION problem asks whether we can transform  $I_0$  into  $I_t$  via independent sets of  $G$ , each of which results from the previous one by either adding or subtracting a single node of  $G$ , without ever going through an independent set of cardinality less than  $k' - 1$ . This problem is known to be PSPACE-complete [7].

We now construct the corresponding instance of SUBSET SUM RECONFIGURATION with conflict graph. The set  $A$  contains  $n$  items, and let  $s(a) = 1$  for all items  $a$  in  $A$ . Each item in  $A$  corresponds to a node of  $G$ , and the conflict graph for  $A$  is connected as  $G$ . The knapsack is of capacity  $c = n$ , and let the threshold  $k = k' - 1$ . Finally, the two packings  $A_0$  and  $A_t$  consist of the items which correspond to the nodes in  $I_0$  and  $I_t$ , respectively; and hence both  $A_0$  and  $A_t$  are of total size at least  $k' = k + 1$ .

Since every feasible packing of total size at least  $k$  induces an independent set in  $G$  of cardinality at least  $k = k' - 1$ , it is obvious that there is a desired transformation between  $I_0$  and  $I_t$  if and only if  $\text{OPT}(A_0, A_t) \geq k$ .  $\square$

We finally have the following inapproximability result, whose proof is omitted due to the page limitation.

**Theorem 3.** MAXMIN SUBSET SUM RECONFIGURATION with conflict graph is APX-hard, and cannot be approximated within any constant factor unless  $P = NP$ .

### 3 PTAS

Since MAXMIN SUBSET SUM RECONFIGURATION with conflict graph is APX-hard, this variant does not admit a PTAS unless  $P = NP$ . However, in this section, we give a PTAS for the original version. Remember that, since we have shown in Theorem 1 that the problem is strongly NP-hard, there is no FPTAS for the problem unless  $P = NP$ ; in this sense, a PTAS is the best approximation algorithm we can expect for the problem. We have the following theorem.

**Theorem 4.** There is a polynomial-time approximation scheme for MAXMIN SUBSET SUM RECONFIGURATION.

In the remainder of this section, as a proof of Theorem 4, we give an algorithm which actually finds a reconfiguration sequence  $\mathcal{P}$  between two given packings  $A_0$  and  $A_t$  such that  $f(\mathcal{P}) \geq (1 - \varepsilon')\text{OPT}(A_0, A_t)$  in time polynomial in  $n$  (but, exponential in  $1/\varepsilon'$ ) for any fixed constant  $\varepsilon'$ ,  $0 < \varepsilon' < 1$ , where  $n$  is the number of items in the set  $A$ . Therefore, our approximate objective value  $\text{APPRO}(A_0, A_t)$  is  $f(\mathcal{P})$ , and hence the error is bounded by  $\varepsilon'\text{OPT}(A_0, A_t)$ , that is,

$$\text{OPT}(A_0, A_t) - \text{APPRO}(A_0, A_t) = \text{OPT}(A_0, A_t) - f(\mathcal{P}) \leq \varepsilon'\text{OPT}(A_0, A_t).$$

As we have mentioned in the Introduction, the placement of items is the central matter in the reconfiguration problem. Therefore, we construct an edge-weighted graph, called a configuration graph, which represents all (feasible) packings together with their adjacency. For a set  $A$  of items and a knapsack of capacity  $c$ ,

a *configuration graph*  $\mathcal{C} = (\mathcal{V}, \mathcal{E})$  is defined as follows: each vertex in  $\mathcal{V}$  corresponds to a packing  $A_i$ , and two vertices are joined by an edge  $e$  in  $\mathcal{E}$  if and only if the corresponding two packings  $A_i$  and  $A_j$  are adjacent; the *weight*  $\omega(e)$  of  $e$  is defined as follows:  $\omega(e) = \min\{s(A_i), s(A_j)\}$ . Notice that the weight  $\omega(e)$  of an edge  $e$  corresponds to the objective value  $f(\mathcal{P}_{i,j})$  for the reconfiguration sequence  $\mathcal{P}_{i,j} = \{A_i, A_j\}$  along  $e$ . Figure 2 illustrates the configuration graph for a set  $A = \{5, 6, 8, 11\}$  and a knapsack of capacity  $c = 20$ , where each vertex is drawn as a box and each edge as a line. From now on, we may call a packing simply a vertex of a configuration graph if it is clear from the context. Since there is a vertex corresponding to the empty packing, a configuration graph is always connected. Then, MAXMIN SUBSET SUM RECONFIGURATION can be seen as the problem of maximizing the minimum edge-weight in a path between  $A_0$  and  $A_t$  in  $\mathcal{C}$ . It is easy to see that the problem can be solved in time polynomial in  $|\mathcal{V}| + |\mathcal{E}|$  by the following naive algorithm: delete all edges having the smallest weight from  $\mathcal{C}$ , and check whether the two vertices  $A_0$  and  $A_t$  remain in the same connected component of the resulting graph; if so, let  $\mathcal{C}$  be the resulting graph and repeat. Note that, however, the size  $|\mathcal{V}| + |\mathcal{E}|$  of  $\mathcal{C}$  can be an exponential in  $n$ .

We now briefly explain our PTAS together with the organization of this section. For a fixed constant  $\varepsilon'$ ,  $0 < \varepsilon' < 1$ , let

$$\varepsilon = \frac{1}{2}\varepsilon'. \quad (1)$$

(The reason why the coefficient above is  $1/2$  will be explained in Section 3.4.) Given a set  $A$  of items and a fixed constant  $\varepsilon$ ,  $0 < \varepsilon < 1/2$ , we divide the items of  $A$  into two groups: an item  $a$  is called a *large item* if  $s(a) \geq \varepsilon c/2$ ; otherwise the item is called a *small item*. We show in Section 3.1 that the problem can be optimally solved in polynomial time if  $A$  contains only large items; in this case, the number of packings (and hence the number of vertices in the configuration graph) can be bounded by a polynomial in  $n$ . In Section 3.2 we then explain that small items can be moved greedily with only small error. In Section 3.3 we finally deal with a general instance by combining the techniques above, without losing the reachability and with keeping the small error. Section 3.4 gives the analysis of our algorithm.

### 3.1 Large Items

In this subsection, we show that MAXMIN SUBSET SUM RECONFIGURATION can be optimally solved in polynomial time if the given set  $A$  contains only large items. It suffices to show that we can construct the corresponding configuration graph  $\mathcal{C} = (\mathcal{V}, \mathcal{E})$  in polynomial time for such an instance, and that the size  $|\mathcal{V}| + |\mathcal{E}|$  of  $\mathcal{C}$  is a polynomial in  $n$ . Formally, we have the following lemma.

**Lemma 1.** *For a fixed constant  $\varepsilon > 0$ , suppose that every item in the set  $A$  is of size at least  $\varepsilon c/2$ , where  $c$  is the capacity of the knapsack. Then, MAXMIN SUBSET SUM RECONFIGURATION can be optimally solved in polynomial time.*

*Proof.* Since  $s(a) \geq \varepsilon c/2$  for each item  $a \in A$ , the number of items in any (feasible) packing is bounded by  $\lfloor 2/\varepsilon \rfloor$ . Let  $\gamma = \lfloor 2/\varepsilon \rfloor$ , then  $\gamma$  is a fixed constant. We denote by  $N$  the number of vertices (packings) in the corresponding configuration graph  $\mathcal{C} = (\mathcal{V}, \mathcal{E})$ , that is,  $N = |\mathcal{V}|$ . Since  $A$  contains  $n$  items and each packing consists of at most  $\gamma$  items, it is easy to see that  $N$  can be bounded by  $\binom{n+\gamma}{\gamma}$ . Therefore,  $N$  is a polynomial in  $n$ , and hence we can construct  $\mathcal{C}$  in time polynomial in  $n$ . Since the size  $|\mathcal{V}| + |\mathcal{E}|$  of  $\mathcal{C}$  is a polynomial in  $n$ , we can solve the problem optimally in polynomial time.  $\square$

### 3.2 Small Items

Suppose in this subsection that the given set  $A$  may contain small items. Then, the number of items in a packing cannot be bounded by a constant, and hence the number  $N = |\mathcal{V}|$  of vertices in the configuration graph  $\mathcal{C} = (\mathcal{V}, \mathcal{E})$  cannot be always bounded by a polynomial in  $n$ ; more specifically,  $N$  can be  $O(2^n)$ . Therefore, we will later (in Section 3.3) construct an “approximate configuration graph  $\mathcal{C}_A$ ,” whose size is bounded by a polynomial in  $n$ .

We now explain how to find a reconfiguration sequence greedily when  $A_0 \Delta A_t$  contains only small items for two given packings  $A_0$  and  $A_t$ . Let  $L_\varepsilon$  be the set of large items in  $A$ , that is,  $L_\varepsilon = \{a \in A \mid s(a) \geq \varepsilon c/2\}$ , and let  $S_\varepsilon = A \setminus L_\varepsilon$ . We have the following lemma.

**Lemma 2.** *Let  $A_0$  and  $A_t$  be an arbitrary pair of packings such that  $A_0 \Delta A_t \subseteq S_\varepsilon$ . Then, there exists a reconfiguration sequence  $\mathcal{P}_s$  between  $A_0$  and  $A_t$  such that*

- (a) *no item in  $L_\varepsilon$  is moved in  $\mathcal{P}_s$ ; and*
- (b)  *$f(\mathcal{P}_s) \geq (1 - \varepsilon) \min\{s(A_0), s(A_t)\}$ .*

*Moreover, such a reconfiguration sequence  $\mathcal{P}_s$  can be found in linear time.*

*Proof.* We give an  $O(n)$ -time algorithm which finds a reconfiguration sequence  $\mathcal{P}_s$  between  $A_0$  and  $A_t$  satisfying (a) and (b), as follows.

**Case (i):**  $s(A_0 \cup A_t) \leq c$ .

In this case, we first add all items in  $A_t \setminus A_0$  one by one, and obtain the packing  $A_0 \cup A_t$ ; and then, delete all items in  $A_0 \setminus A_t$  one by one, and obtain  $A_t$ . Note that  $A_t \setminus A_0 \subseteq A_0 \Delta A_t \subseteq S_\varepsilon$  and  $A_0 \setminus A_t \subseteq A_0 \Delta A_t \subseteq S_\varepsilon$ , and hence no item in  $L_\varepsilon$  is moved in this reconfiguration sequence  $\mathcal{P}_s$ . We clearly have

$$f(\mathcal{P}_s) = \min\{s(A_0), s(A_t)\} > (1 - \varepsilon) \min\{s(A_0), s(A_t)\}.$$

Therefore,  $\mathcal{P}_s$  satisfies both (a) and (b). Moreover,  $\mathcal{P}_s$  can be found in linear time since we move each item in  $A_0 \Delta A_t$  only once.

**Case (ii):**  $s(A_0 \cup A_t) > c$ .

In this case, we first add items in  $A_t \setminus A_0$  one by one in arbitrary order as many as possible; let  $A_j$  be the current packing. Then,  $s(A_j) > (1 - \frac{\varepsilon}{2})c$  because, otherwise, we can add more items to  $A_j$  since  $s(a) < \varepsilon c/2$  for all items  $a \in A_t \setminus A_0$ .

We then delete items in  $A_0 \setminus A_t$  one by one in arbitrary order until we obtain a packing  $A'_j$  such that

$$(1 - \varepsilon)c < s(A'_j) \leq \left(1 - \frac{\varepsilon}{2}\right)c. \quad (2)$$

Since  $s(a) < \varepsilon c/2$  for all items  $a \in A_0 \setminus A_t$ , we can always find such a packing  $A'_j$ . If  $s(A'_j \cup A_t) \leq c$ , then go to Case (i) above; otherwise, repeat Case (ii). Note that, in this reconfiguration sequence  $\mathcal{P}_s$ , every addition is executed for an item in  $A_t \setminus A_0$  ( $\subseteq S_\varepsilon$ ) and every deletion is done for an item in  $A_0 \setminus A_t$  ( $\subseteq S_\varepsilon$ ). Thus,  $\mathcal{P}_s$  satisfies (a). Furthermore, since each item in  $A_0 \Delta A_t$  is moved exactly once,  $\mathcal{P}_s$  can be found in linear time. We now show that (b) holds for  $\mathcal{P}_s$ . By Eq. (2) we have

$$f(\mathcal{P}_s) \geq \min\left\{(1 - \varepsilon)c, \min\{s(A_0), s(A_t)\}\right\}.$$

Since  $c \geq \min\{s(A_0), s(A_t)\}$ , we have  $f(\mathcal{P}_s) \geq (1 - \varepsilon) \min\{s(A_0), s(A_t)\}$ .  $\square$

### 3.3 General Instance

We finally deal with a general instance, that is, a set  $A$  may contain small items and two packings  $A_0$  and  $A_t$  do not necessarily satisfy  $A_0 \Delta A_t \subseteq S_\varepsilon$ . Our idea is to construct an *approximate configuration graph*  $\mathcal{C}_A$ , as follows.

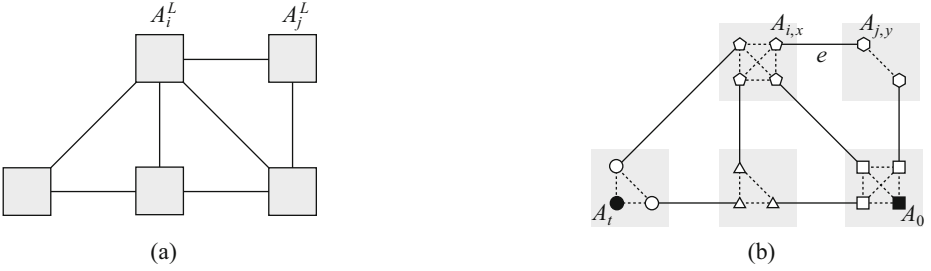
#### Step 1: Configuration graph for $L_\varepsilon$

We first construct a configuration graph  $\mathcal{C}_{L_\varepsilon} = (\mathcal{V}_{L_\varepsilon}, \mathcal{E}_{L_\varepsilon})$  for the large item set  $L_\varepsilon$  of  $A$  and the capacity  $c$ . Then, as in Lemma 1,  $\mathcal{C}_{L_\varepsilon}$  can be constructed in time polynomial in  $n$ , and the size  $|\mathcal{V}_{L_\varepsilon}| + |\mathcal{E}_{L_\varepsilon}|$  of  $\mathcal{C}_{L_\varepsilon}$  can be bounded by a polynomial in  $n$ . Figure 3(a) illustrates the configuration graph for  $L_\varepsilon$  of  $A$ , where each box corresponds to a packing consisting of only large items. Note that  $\mathcal{C}_{L_\varepsilon}$  contains the vertex corresponding to the empty packing, and hence  $\mathcal{C}_{L_\varepsilon}$  is connected.

#### Step 2: Small items

We then expand  $\mathcal{C}_{L_\varepsilon}$  into the approximate configuration graph  $\mathcal{C}_A = (\mathcal{V}_A, \mathcal{E}_A)$ , as illustrated in Fig.3(b). For each edge in  $\mathcal{C}_{L_\varepsilon}$  joining two vertices  $A_i^L$  and  $A_j^L$  (that consist only of large items), we replace it with an edge  $e$  that joins two new vertices  $A_{i,x}$  and  $A_{j,y}$ , called *gate vertices* or *gate packings*, defined as follows. Assume without loss of generality that  $A_j^L = A_i^L \cup \{a\}$  for some large item  $a$  in  $L_\varepsilon$ , and hence  $A_j^L$  can be obtained by adding one large item  $a$  to  $A_i^L$ . To extend  $A_j^L$  to the gate packing  $A_{j,y}$  containing small items, we find a packing  $A_j^S \subseteq S_\varepsilon$  of small items for the remaining space  $c - s(A_j^L)$  of the knapsack; we employ an FPTAS for the ordinary MAXIMUM SUBSET SUM problem [9] for the fixed constant  $\varepsilon$ . Then, let  $A_{j,y} = A_j^L \cup A_j^S$  and let  $A_{i,x} = A_{j,y} \setminus \{a\}$ . Note that  $A_{i,x} \Delta A_{j,y} = \{a\}$  and hence  $A_{i,x}$  and  $A_{j,y}$  are adjacent. We call the edge  $e = (A_{i,x}, A_{j,y})$  an *external edge*, and the weight  $\omega(e)$  is defined as follows:  $\omega(e) = \min\{s(A_{i,x}), s(A_{j,y})\} = s(A_{i,x})$ . In Fig.3(b), each gate packing





**Fig. 3.** (a) Configuration graph  $C_{L_\epsilon}$  for the large item set  $L_\epsilon$  of  $A$ , and (b) approximate configuration graph  $C_A$  for  $A$

is represented by a circle, triangle, square, pentagon, or hexagon, colored with white; all gate packings represented by the same symbol have the same placement of large items; and each external edge is drawn as a (non-dotted) line.

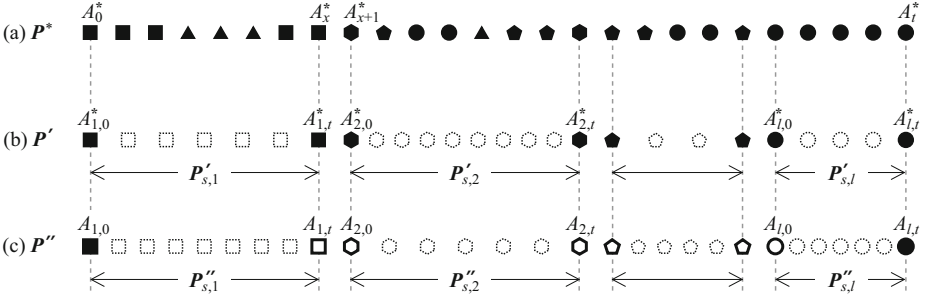
For each vertex  $A_i^L$  in  $C_{L_\epsilon}$ , we have thus created the number  $d(A_i^L)$  of new gate vertices  $A_{i,1}, A_{i,2}, \dots, A_{i,d(A_i^L)}$ , where  $d(A_i^L)$  is the degree of  $A_i^L$  in  $C_{L_\epsilon}$ . Clearly,  $A_{i,x} \cap L_\epsilon = A_{i,z} \cap L_\epsilon$  for every pair of vertices. The original vertex  $A_i^L$  is deleted, and we connect the  $d(A_i^L)$  gate vertices so that they form a clique; for each pair of vertices  $A_{i,x}$  and  $A_{i,z}$ , the edge joining them is called an *internal edge*; in Fig.3(b), each internal edge is drawn as a dotted line. It should be noted that  $A_{i,x}$  and  $A_{i,z}$  are not necessarily adjacent although they are joined by an internal edge. However, using Lemma 2 we can regard such an internal edge as a reconfiguration sequence  $\mathcal{P}_s$  between  $A_{i,x}$  and  $A_{i,z}$  such that  $f(\mathcal{P}_s) \geq (1 - \epsilon) \min\{s(A_{i,x}), s(A_{i,z})\}$ . Therefore, the weight  $\omega(e)$  of  $e$  is defined as follows:

$$\omega(e) = \begin{cases} \min\{s(A_{i,x}), s(A_{i,z})\} & \text{if } A_{i,x} \text{ and } A_{i,z} \text{ are adjacent;} \\ (1 - \epsilon) \min\{s(A_{i,x}), s(A_{i,z})\} & \text{otherwise.} \end{cases} \quad (3)$$

**Step 3:  $A_0$  and  $A_t$**

The current graph above does not always contain the vertices corresponding to given packings  $A_0$  and  $A_t$ . If the graph does not contain  $A_0$ , then we add a new vertex  $A_0$  to the graph, and join it with each gate vertex having the same placement  $A_0 \cap L_\epsilon$  of large items by an internal edge. (The case for  $A_t$  is similar.) This completes the construction of the approximate configuration graph  $C_A = (\mathcal{V}_A, \mathcal{E}_A)$ .

Clearly, a path between the two vertices  $A_0$  and  $A_t$  in  $C_A$  corresponds to a reconfiguration sequence between the two packings  $A_0$  and  $A_t$ . Since  $|\mathcal{V}_A| \leq 2|\mathcal{E}_{L_\epsilon}| + 2$  and  $|\mathcal{E}_{L_\epsilon}|$  is bounded by a polynomial in  $n$ , the size  $|\mathcal{V}_A| + |\mathcal{E}_A|$  of  $C_A$  is bounded by a polynomial in  $n$ . Therefore, we can find in polynomial time a path between  $A_0$  and  $A_t$  whose minimum edge-weight is maximum in  $C_A$ ; we choose the corresponding reconfiguration sequence  $\mathcal{P}$  as our approximate solution.



**Fig. 4.** Reconfiguration sequences  $\mathcal{P}^*$ ,  $\mathcal{P}'$  and  $\mathcal{P}''$  between  $A_0$  and  $A_t$

### 3.4 Analysis of the Algorithm

We have shown in Section 3.3 that our algorithm finds a reconfiguration sequence  $\mathcal{P}$  between  $A_0$  and  $A_t$  in polynomial time. In this subsection, we show that  $\mathcal{P}$  satisfies  $f(\mathcal{P}) \geq (1 - \varepsilon')\text{OPT}(A_0, A_t)$  for a fixed constant  $\varepsilon'$ ,  $0 < \varepsilon' < 1$ , as required.

Let  $\mathcal{P}^* = \{A_0^*, A_1^*, \dots, A_t^*\}$  be an arbitrary optimal reconfiguration sequence between  $A_0$  and  $A_t$ , where  $A_0^* = A_0$  and  $A_t^* = A_t$ . Figure 4(a) illustrates the optimal reconfiguration sequence  $\mathcal{P}^*$ , where each black symbol corresponds to a packing  $A_i^*$  in  $\mathcal{P}^*$ , and all packings represented by the same symbol have the same placement of large items. Let  $A_{\min}^*$  be a packing in  $\mathcal{P}^*$  whose total size is minimum, and hence  $f(\mathcal{P}^*) = s(A_{\min}^*)$ . Then, we have

$$s(A_{\min}^*) = \text{OPT}(A_0, A_t), \quad (4)$$

and

$$s(A_i^*) \geq s(A_{\min}^*) \quad (5)$$

for each packing  $A_i^*$ ,  $0 \leq i \leq t$ .

From now on, we transform  $\mathcal{P}^*$  into another reconfiguration sequence  $\mathcal{P}''$  between  $A_0$  and  $A_t$  so that  $\mathcal{C}_A$  contains the path corresponding to  $\mathcal{P}''$ . Remember that our algorithm finds a reconfiguration sequence  $\mathcal{P}$  between  $A_0$  and  $A_t$  which is optimal in  $\mathcal{C}_A$ , and hence we have

$$\text{APPRO}(A_0, A_t) = f(\mathcal{P}) \geq f(\mathcal{P}''). \quad (6)$$

We first transform  $\mathcal{P}^*$  into a reconfiguration sequence  $\mathcal{P}'$  between  $A_0$  and  $A_t$  such that the same placement of large items appears consecutively. This can be done by the following algorithm: find the last packing  $A_x^*$  in  $\mathcal{P}^*$  such that  $A_x^* \cap L_\varepsilon = A_0^* \cap L_\varepsilon$ ; replace the subsequence  $\{A_0^*, A_1^*, \dots, A_x^*\}$  with a reconfiguration sequence  $\mathcal{P}_s$  between  $A_0^*$  and  $A_x^*$  obtained by Lemma 2; set  $A_0^* = A_{x+1}^*$  and repeat. We denote by  $\mathcal{P}'_{s,i}$  the reconfiguration (sub)sequence obtained by the  $i$ th step of the algorithm above. (See Fig.4(b), where the intermediate packings in  $\mathcal{P}'_{s,i}$  are represented by dotted symbols.) By Lemma 2(a) all packings in  $\mathcal{P}'_{s,i}$  have the same placement of large items, and hence all intermediate packings in  $\mathcal{P}'_{s,i}$

are depicted by the same symbol in Fig.4(b). Moreover, from the construction, every two reconfiguration subsequences  $\mathcal{P}'_{s,i}$  and  $\mathcal{P}'_{s,j}$  with  $i \neq j$  have different placements of large items. Let  $\ell$  be the number of iterations of the algorithm, and hence  $\mathcal{P}' = \mathcal{P}'_{s,1} \cup \mathcal{P}'_{s,2} \cup \dots \cup \mathcal{P}'_{s,\ell}$ . For each  $i$ ,  $1 \leq i \leq \ell$ , let  $A_{i,0}^*$  and  $A_{i,t}^*$  be the first and last packings in  $\mathcal{P}'_{s,i}$ , respectively. Then, from the construction, the optimal reconfiguration sequence  $\mathcal{P}^*$  contains the packings  $A_{i,0}^*$  and  $A_{i,t}^*$ ,  $1 \leq i \leq \ell$ , and hence they are depicted by black symbols in Fig.4(b). On the other hand, intermediate packings are not necessarily contained in  $\mathcal{P}^*$ , and hence they are depicted by white dotted symbols. Note that  $A_{1,0}^* = A_0$  and  $A_{\ell,t}^* = A_t$ .

We then transform  $\mathcal{P}'$  into the reconfiguration sequence  $\mathcal{P}''$  between  $A_0$  and  $A_t$  such that  $\mathcal{C}_A$  contains the path corresponding to  $\mathcal{P}''$ . For each  $i$ ,  $1 \leq i \leq \ell-1$ , two packings  $A_{i,t}^*$  and  $A_{i+1,0}^*$  are adjacent; moreover, the item moved between them is a large item  $a \in L_\varepsilon$ . Therefore,  $\mathcal{C}_A$  contains the external edge  $e = (A_{i,t}, A_{i+1,0})$  which corresponds to moving the item  $a$  from the large item placement  $A_{i,t}^* \cap L_\varepsilon$  to  $A_{i+1,0}^* \cap L_\varepsilon$ . We may regard that the two endpoints (gate packings)  $A_{i,t}$  and  $A_{i+1,0}$  of  $e$  correspond to  $A_{i,t}^*$  and  $A_{i+1,0}^*$ , respectively. Of course, the gate packings  $A_{i,t}$  and  $A_{i+1,0}$  are not always the same as  $A_{i,t}^*$  and  $A_{i+1,0}^*$ , respectively, and hence they are depicted by (non-dotted) white symbols in Fig.4(c). However, it should be noted that  $A_{i,t}^* \cap L_\varepsilon = A_{i,t} \cap L_\varepsilon$  and  $A_{i+1,0}^* \cap L_\varepsilon = A_{i+1,0} \cap L_\varepsilon$ , and hence  $A_{i,t}$  and  $A_{i+1,0}$  in Fig.4(c) are depicted by the same symbols as  $A_{i,t}^*$  and  $A_{i+1,0}^*$ , respectively. For the sake of notational convenience, let  $A_{1,0} = A_{1,0}^* = A_0$  and  $A_{\ell,t} = A_{\ell,t}^* = A_t$ . Since  $A_{i,0}$  and  $A_{i,t}$ ,  $1 \leq i \leq \ell$ , have the same large item placement and are contained in  $\mathcal{C}_A$ , there exists the internal edge  $e_{s,i}$  joining them in  $\mathcal{C}_A$ ; let  $\mathcal{P}''_{s,i}$  be the reconfiguration subsequence between  $A_{i,0}$  and  $A_{i,t}$  corresponding to  $e_{s,i}$ . By Eq. (3) we have

$$f(\mathcal{P}''_{s,i}) = \omega(e_{s,i}) \geq (1 - \varepsilon) \min\{s(A_{i,0}), s(A_{i,t})\} \tag{7}$$

for each  $i$ ,  $1 \leq i \leq \ell$ . Let  $\mathcal{P}'' = \mathcal{P}''_{s,1} \cup \mathcal{P}''_{s,2} \cup \dots \cup \mathcal{P}''_{s,\ell}$ , then  $f(\mathcal{P}'') = \min\{f(\mathcal{P}''_{s,i}) : 1 \leq i \leq \ell\}$ . This completes the construction of  $\mathcal{P}''$ .

We now show the following lemma, whose proof is omitted due to the page limitation.

**Lemma 3.**  $s(A_{i,0}) > (1 - \varepsilon)s(A_{i,0}^*)$  and  $s(A_{i,t}) > (1 - \varepsilon)s(A_{i,t}^*)$  for each  $i$ ,  $1 \leq i \leq \ell$ .

Assume that  $\mathcal{P}''_{s,k}$  contains the packing whose total size is minimum in  $\mathcal{P}''$ . Then, by Eq. (7) we have  $f(\mathcal{P}'') = f(\mathcal{P}''_{s,k}) \geq (1 - \varepsilon) \min\{s(A_{k,0}), s(A_{k,t})\}$ . Therefore, by Lemma 3 and Eqs. (4) and (5) we have

$$\begin{aligned} f(\mathcal{P}'') &> (1 - \varepsilon)^2 \min\{s(A_{k,0}^*), s(A_{k,t}^*)\} \geq (1 - \varepsilon)^2 s(A_{\min}^*) \\ &> (1 - 2\varepsilon)s(A_{\min}^*) = (1 - 2\varepsilon)\text{OPT}(A_0, A_t). \end{aligned} \tag{8}$$

By Eqs. (1), (6) and (8) we have

$$\text{APPRO}(A_0, A_t) \geq f(\mathcal{P}'') > (1 - 2\varepsilon)\text{OPT}(A_0, A_t) = (1 - \varepsilon')\text{OPT}(A_0, A_t).$$

This completes the proof of Theorem 4. □

## 4 Concluding Remark

In the ordinary KNAPSACK problem [4,9], each item is assigned not only a size but also a *profit*, and we wish to find a packing whose total profit is at least a given threshold. Consider the two reconfiguration problems for KNAPSACK, called KNAPSACK RECONFIGURATION and MAXMIN KNAPSACK RECONFIGURATION, which are defined similarly as SUBSET SUM RECONFIGURATION and MAXMIN SUBSET SUM RECONFIGURATION, respectively. Because they are generalizations of our reconfiguration problems for SUBSET SUM, the complexity and inapproximability results in Section 2 hold also for them. However, it remains open to obtain a PTAS for MAXMIN KNAPSACK RECONFIGURATION.

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