# Approximability of Partitioning Graphs with Supply and Demand


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**Abstract**

Suppose that each vertex of a graph $G$ is either a supply vertex or a demand vertex and is assigned a positive real number, called the supply or the demand. Each demand vertex can receive "power" from at most one supply vertex through edges in $G$. One thus wishes to partition $G$ into connected components by deleting edges from $G$ so that each component $C$ either has no supply vertex or has exactly one supply vertex whose supply is at least the sum of demands in $C$, and wishes to maximize the fulfillment, that is, the sum of demands in all components with supply vertices. This maximization problem is known to be NP-hard even for trees having exactly one supply vertex and strongly NP-hard for general graphs. In this paper, we focus on the approximability of the problem. We first show that the problem is MAXSNP-hard and hence there is no polynomial-time approximation scheme (PTAS) for general graphs unless P = NP. We then present a fully polynomial-time approximation scheme (FPTAS) for series-parallel graphs having exactly one supply vertex.


## 1 Introduction

Consider a graph $G$ such that each vertex is either a supply vertex or a demand vertex. Each vertex $v$ is assigned a positive real number; the number


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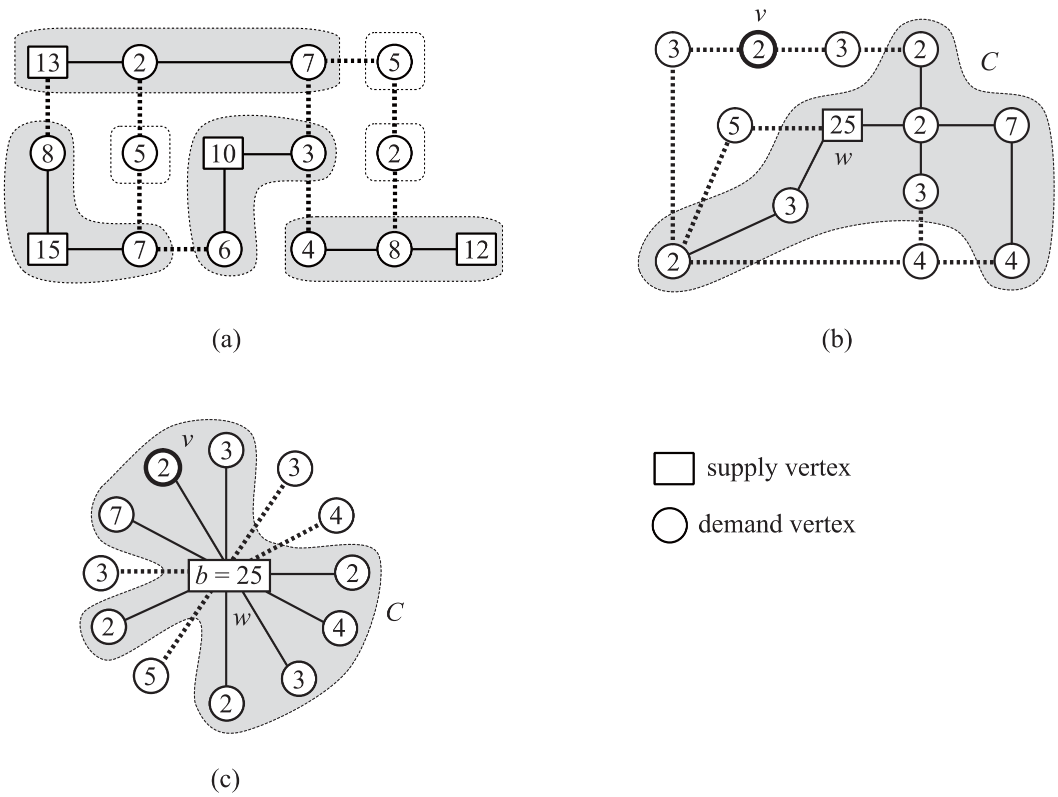



Fig. 1. (a) Partition of a graph with maximum fulfillment, (b) partition of a series–parallel graph $G$ having exactly one supply vertex, and (c) a star $S$ with a supply vertex at the center.

is called the *supply of* $v$ if $v$ is a supply vertex; otherwise, it is called the *demand of* $v$. Each demand vertex can receive "power" from at most one supply vertex through edges in $G$. One thus wishes to partition $G$ into connected components by deleting edges from $G$ so that each component $C$ has exactly one supply vertex whose supply is at least the sum of demands of all demand vertices in $C$. However, such a partition does not always exist. So we wish to obtain a *partition* of $G$ into connected components so that each component $C$ either has no supply vertex or has exactly one supply vertex whose supply is at least the sum of demands of all demand vertices in $C$, and wish to maximize the "fulfillment," that is, the sum of demands of the demand vertices in all components with supply vertices. We call this problem the *maximum partition problem* [7]. Figure 1(a) illustrates a solution of the maximum partition problem for a graph, whose fulfillment is $(2+7)+(8+7)+(3+6)+(4+8) = 45$. In Fig. 1(a) each supply vertex is drawn as a rectangle and each demand vertex as a circle, the supply or demand is written inside, the deleted edges are drawn by thick dotted lines, and each connected component with a supply vertex is shaded.

The maximum partition problem has some applications to the power supply problem for power delivery networks [3,7,10,14]. Let $G$ be a graph of a power delivery network. Each supply vertex represents a “feeder,” which can supply electrical power. Each demand vertex represents a “load,” which requires electrical power supplied from exactly one of the feeders through a network. Each edge of $G$ represents a cable segment, which can be “turned off” by a switch. Then the maximum partition problem represents the “power supply switching problem” to maximize the sum of all loads that can be supplied powers in a network “reconfigured” by turning off some cable segments.

Given a set $A$ of integers and an upper bound (integer) $b$, the *maximum subset sum problem* [4,5] asks to find a subset $C$ of $A$ such that the sum of integers in $C$ is no greater than the bound $b$ and is maximum among all such subsets $C$. The maximum subset sum problem can be reduced in linear time to the maximum partition problem for a particular tree, called a star, with exactly one supply vertex at the center, as illustrated in Fig. 1(c) [7]. Since the maximum subset sum problem is NP-hard, the maximum partition problem is also NP-hard even for stars. Thus it is very unlikely that the maximum partition problem can be exactly solved in polynomial time even for trees. However, there is a fully polynomial-time approximation scheme (FPTAS) for the maximum partition problem on trees [7]. One may thus expect that the FPTAS for trees can be extended to a larger class of graphs, for example series-parallel graphs and partial $k$-trees, that is, graphs with bounded treewidth [1,2].

In this paper, we study the approximability of the maximum partition problem. We first show that the maximum partition problem is MAXSNP-hard, and hence there is no polynomial-time approximation scheme (PTAS) for the problem on general graphs unless P = NP. We then present an FPTAS for series-parallel graphs having exactly one supply vertex. The FPTAS for series-parallel graphs can be extended to partial $k$-trees. Figure 1(b) depicts a series-parallel graph together with a connected component $C$ found by our FPTAS. One might think that it would be straightforward to extend the FPTAS for the maximum subset sum problem in [5] to an FPTAS for the maximum partition problem with a single supply vertex. However, this is not the case since we must take a graph structure into account. For example, the vertex $v$ of demand 2 drawn by a thick circle in Fig. 1(b) cannot be supplied power even though the supply vertex $w$ has marginal power $25-(2+3+2+2+3+7+4) = 2$, while the vertex $v$ in Fig. 1(c) can be supplied power from the supply vertex $w$ in the star having the same supply and demands as in Fig. 1(b). Indeed, we not only extend the “scaling and rounding” technique but also employ many new ideas to derive our FPTAS. An early version of the paper has been presented at [6].

The rest of the paper is organized as follows. In Section 2 we show that the maximum partition problem is MAXSNP-hard. In Section 3 we present a pseudo-polynomial-time algorithm for series-parallel graphs. In Section 4 we present an FPTAS based on the algorithm in Section 3.

## 2 MAXSNP-hardness

Assume in this section that a graph $G$ has one or more supply vertices. (See Figs. 1(a) and 2(b).) The main result of this section is the following theorem.

**Theorem 1** *The maximum partition problem is MAXSNP-hard for bipartite graphs.*

A variant of the MAXSAT problem, called the "3-occurrence MAX3SAT problem," is MAXSNP-hard [11,12]. An instance $\Phi$ of the problem consists of a collection of $m$ clauses $C_1, C_2, \cdots, C_m$ on $n$ variables $x_1, x_2, \cdots, x_n$ such that each clause has exactly three literals and each variable appears at most three times in the clauses. The *3-occurrence MAX3SAT* problem is to find a truth assignment for the variables which satisfies the maximum number of clauses. Since each clause has exactly three literals, we have

$$n \leq 3m. \tag{1}$$

In order to prove Theorem 1, we use the concept of "L-reduction" which is a special kind of reduction that preserves approximability [11,12]. Suppose that both $A$ and $B$ are maximization problems. Then we say that $A$ *can be L-reduced to* $B$ if there exist two polynomial-time algorithms $Q$ and $R$ and two positive constants $\alpha$ and $\beta$ which satisfy the following two conditions (1) and (2) for each instance $I_A$ of $A$:

(1) the algorithm $Q$ returns an instance $I_B = Q(I_A)$ of $B$ such that

$$OPT_B(I_B) \leq \alpha \cdot OPT_A(I_A),$$

where $OPT_A(I_A)$ and $OPT_B(I_B)$ denote the maximum solution values of $I_A$ and $I_B$, respectively; and

(2) for each feasible solution of $I_B$ with value $c_B$, the algorithm $R$ returns a feasible solution of $I_A$ with value $c_A$ such that

$$OPT_A(I_A) - c_A \leq \beta \cdot (OPT_B(I_B) - c_B).$$

Note that, by condition (2) of the L-reduction, $R$ must return the optimal solution of $I_A$ for the optimal solution of $I_B$.

We now prove Theorem 1.

**Proof of Theorem 1.**

It suffices to show that the 3-occurrence MAX3SAT problem can be L-reduced to the maximum partition problem for bipartite graphs.

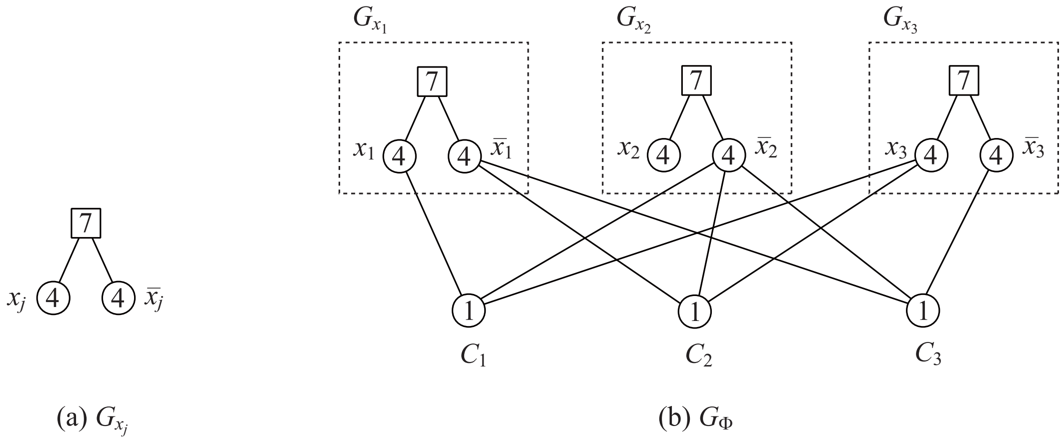



Fig. 2. (a) Variable gadget $G_{x_j}$, and (b) bipartite graph $G_\Phi$ corresponding to an instance $\Phi$ with three clauses $C_1 = (x_1 \vee \bar{x}_2 \vee x_3)$, $C_2 = (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$ and $C_3 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$.

We first show that condition (1) of the L-reduction holds for $\alpha = 26$. It suffices to show that, from each instance $\Phi$ of the 3-occurrence MAX3SAT problem, one can construct in polynomial time a bipartite graph $G_\Phi$ as an instance of the maximum partition problem such that

$$OPT_{MPP}(G_\Phi) \leq 26 \cdot OPT_{SAT}(\Phi), \tag{2}$$

where $OPT_{MPP}(G_\Phi)$ is the maximum solution value of the maximum partition problem for $G_\Phi$ and $OPT_{SAT}(\Phi)$ is the maximum solution value of the 3-occurrence MAX3SAT problem for $\Phi$.

We first make a *variable gadget* $G_{x_j}$ for each variable $x_j$, $1 \leq j \leq n$; $G_{x_j}$ is a binary tree with three vertices as illustrated in Fig. 2(a); the root is a supply vertex of supply 7, and the two leaves $x_j$ and $\bar{x}_j$ are demand vertices of demands 4. Then the graph $G_\Phi$ is constructed as follows. For each variable $x_j$, $1 \leq j \leq n$, put the variable gadget $G_{x_j}$ to the graph, and for each clause $C_i$, $1 \leq i \leq m$, put a demand vertex $C_i$ of demand 1 to the graph. Finally, for each clause $C_i$, $1 \leq i \leq m$, join a demand vertex $x_j$ (or $\bar{x}_j$) in $G_{x_j}$, $1 \leq j \leq n$, with the demand vertex $C_i$ if and only if the literal $x_j$ (or $\bar{x}_j$) is in $C_i$, as illustrated in Fig. 2(b). Clearly, $G_\Phi$ can be constructed in polynomial time, and is a bipartite graph. It should be noted that, since each variable $x_j$, $1 \leq j \leq n$, appears at most three times in the clauses, the supply vertex in $G_{x_j}$ has enough "power" to supply all demand vertices $C_i$ whose corresponding clauses have $x_j$ or $\bar{x}_j$.

We then verify Eq. (2). One can easily have

$$OPT_{MPP}(G_\Phi) = 4n + OPT_{SAT}(\Phi). \tag{3}$$

Note that, for each $j$, $1 \le j \le n$, exactly one of the two demand vertices $x_j$ and $\bar{x}_j$ is supplied power in the maximum solution of the maximum partition problem for $G_\Phi$ and hence the first term $4n$ of the right side of Eq. (3) represents the sum of the demands in $G_{x_j}$, $1 \le j \le n$, which are supplied power. Since $OPT_{SAT}(\Phi) \le m$, by Eqs. (1) and (3) we have

$$OPT_{MPP}(G_\Phi) \le 12m + m = 13m. \tag{4}$$

On the other hand, we have $OPT_{SAT}(\Phi) \ge m/2$, because if a truth assignment satisfies only less than half of clauses of $\Phi$, then the negation of the truth assignment satisfies at least half of the clauses of $\Phi$. Therefore, by Eq. (4) we have

$$OPT_{MPP}(G_\Phi) \le 13m = 26 \cdot \frac{m}{2} \le 26 \cdot OPT_{SAT}(\Phi).$$

We have thus verified Eq. (2).

We next show that condition (2) of the L-reduction holds for $\beta = 1$. One can give a truth assignment in polynomial time from a partition $P$ of $G_\Phi$, as follows: set a variable $x_j$ to TRUE if the demand vertex $x_j$ in $G_{x_j}$ is supplied power in $P$; otherwise, set $x_j$ to FALSE. It suffices to show that

$$OPT_{SAT}(\Phi) - c_\Phi \le OPT_{MPP}(G_\Phi) - f(P), \tag{5}$$

where $c_\Phi$ is the number of clauses of $\Phi$ satisfied by the truth assignment and $f(P)$ is the fulfillment of $P$, that is, the sum of demands of all demand vertices in components with supply vertices. One can easily observe

$$c_\Phi \ge f(P) - 4n. \tag{6}$$

Note that both of the two demand vertices $x_j$ and $\bar{x}_j$ may not be supplied power in $P$ for some variable gadgets $G_{x_j}$. By Eqs. (3) and (6) we have

$$\begin{aligned} OPT_{SAT}(\Phi) - c_\Phi &\le OPT_{SAT}(\Phi) - (f(P) - 4n) \\ &= (4n + OPT_{SAT}(\Phi)) - f(P) \\ &= OPT_{MPP}(G_\Phi) - f(P). \end{aligned}$$

We have thus verified Eq. (5). $\square$

## 3 Pseudo-polynomial-time algorithm

Since the maximum partition problem is strongly NP-hard [8], there is no pseudo-polynomial-time algorithm for general graphs unless P = NP. However, Ito *et al.* presented a pseudo-polynomial-time algorithm for the maximum partition problem on series-parallel graphs having one or more supply vertices [8]. In this section we present another pseudo-polynomial-time algorithm on series-parallel graphs having exactly one supply vertex, which is suited to an FPTAS presented in Section 4. More precisely, we have the following theorem.

**Theorem 2** *The maximum partition problem for a series-parallel graph $G$ with a single supply vertex can be solved in time $O(F^2n)$ if the demands and the supply are integers, where $n$ is the number of vertices in $G$ and $F$ is an arbitrary upper bound on the maximum solution value for $G$.*

A trivial example of the upper bound $F$ is the supply of the supply vertex. Another example is the sum of demands of all demand vertices in $G$. A better upper bound will be given in Section 4.

In the remainder of this section we give an algorithm to solve the maximum partition problem in time $O(F^2n)$ as a proof of Theorem 2. In Subsection 3.1 we give a definition of a series-parallel graph. In Subsection 3.2 we define some terms and present ideas of our algorithm. We then present our algorithm in Subsection 3.3. We finally show, in Subsection 3.4, that our algorithm takes time $O(F^2n)$.

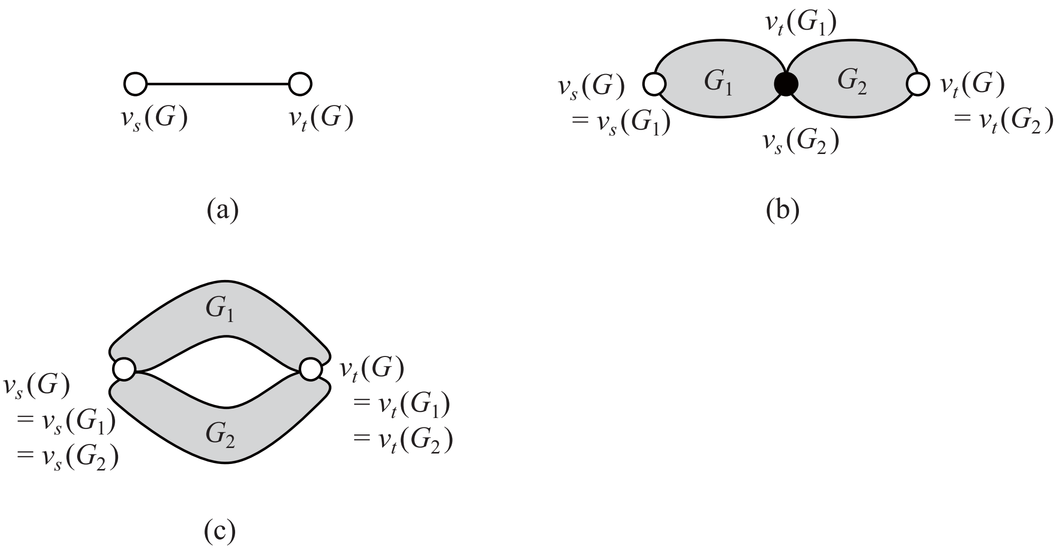



Fig. 3. (a) A series-parallel graph with a single edge, (b) series connection, and (c) parallel connection.

A (*two-terminal*) *series-parallel graph* is defined recursively as follows [13]:

(1) A graph $G$ with a single edge is a series-parallel graph. The ends of the edge are called the *terminals* of $G$ and denoted by $v_s(G)$ and $v_t(G)$. (See Fig. 3(a).)

(2) Let $G_1$ be a series-parallel graph with terminals $v_s(G_1)$ and $v_t(G_1)$, and let $G_2$ be a series-parallel graph with terminals $v_s(G_2)$ and $v_t(G_2)$.

  (a) A graph $G$ obtained from $G_1$ and $G_2$ by identifying $v_t(G_1)$ with $v_s(G_2)$ is a series-parallel graph, whose terminals are $v_s(G) = v_s(G_1)$ and $v_t(G) = v_t(G_2)$. Such a connection is called a *series connection*, and $G$ is denoted by $G = G_1 \bullet G_2$. (See Fig. 3(b).)

  (b) A graph $G$ obtained from $G_1$ and $G_2$ by identifying $v_s(G_1)$ with $v_s(G_2)$ and identifying $v_t(G_1)$ with $v_t(G_2)$ is a series-parallel graph, whose terminals are $v_s(G) = v_s(G_1) = v_s(G_2)$ and $v_t(G) = v_t(G_1) = v_t(G_2)$. Such a connection is called a *parallel connection*, and $G$ is denoted by $G = G_1 \parallel G_2$. (See Fig. 3(c).)

The terminals $v_s(G)$ and $v_t(G)$ of $G$ are often denoted simply by $v_s$ and $v_t$, respectively. Since we deal with the maximum partition problem, we may assume without loss of generality that $G$ is a simple graph and hence $G$ has no multiple edges.

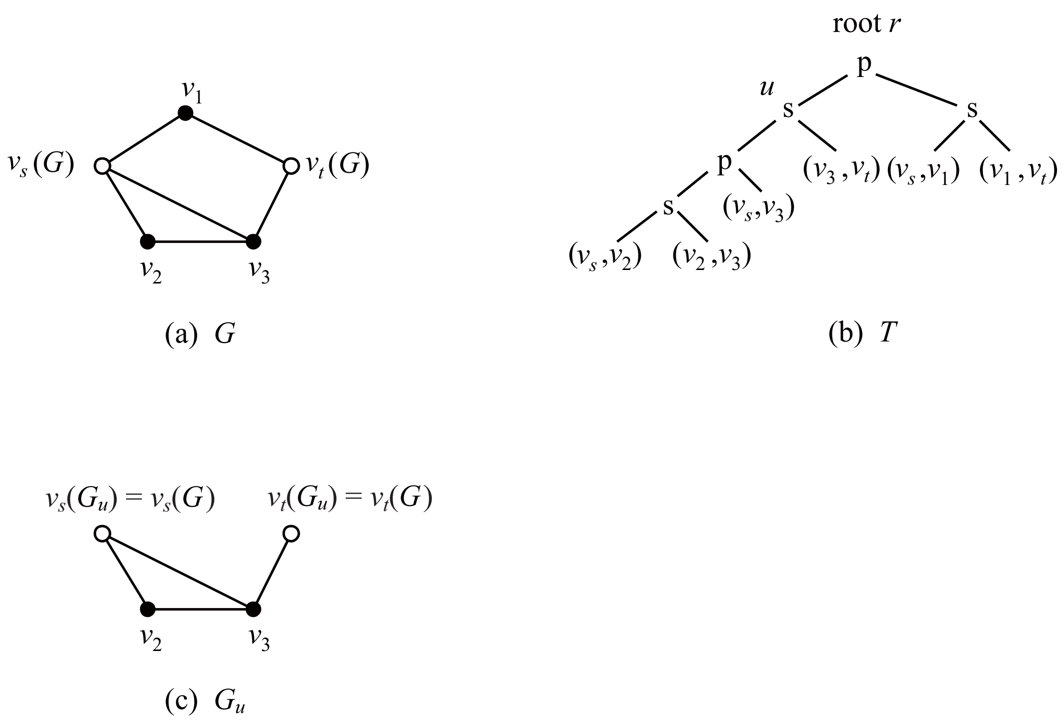



Fig. 4. (a) A series-parallel graph $G$, (b) a binary decomposition tree $T$ of $G$, and (c) a subgraph $G_u$ of $G$.

A series-parallel graph $G$ can be represented by a "binary decomposition tree" [13]. Figure 4(a) illustrates a series-parallel graph $G$, and Fig. 4(b) depicts a binary decomposition tree $T$ of $G$. Labels s and p attached to internal nodes in $T$ indicate series and parallel connections, respectively. Nodes labeled s and p are called *s-* and *p-nodes*, respectively. Every leaf of $T$ represents a subgraph of $G$ induced by a single edge. Each node $u$ of $T$ corresponds to a subgraph $G_u = (V_u, E_u)$ of $G$ induced by all edges represented by the leaves that are descendants of $u$ in $T$. Figure 4(c) depicts $G_u$ for the left child $u$ of the root $r$ of $T$ in Fig. 4(b). $G_u$ is a series-parallel graph for each node $u$ of $T$, and $G = G_r$ for the root $r$ of $T$. Since a binary decomposition tree of a given series-parallel graph $G$ can be found in linear time [13], we may assume that a series-parallel graph $G$ and its binary decomposition tree $T$ are given.

### *3.2 Terms and ideas*

Suppose that there is exactly one supply vertex $w$ in a graph $G = (V, E)$, as illustrated in Figs. 1(b) and (c). Let $\sup(w)$ be the supply of $w$. For each demand vertex $v$, we denote by $\mathrm{dem}(v)$ the demand of $v$. Let $\mathrm{dem}(w) = 0$ although $w$ is a supply vertex. Then, instead of finding a partition of $G$, we shall find a set $C \subseteq V$ such that

(a) $w \in C$;
(b) $\sum_{v \in C} \mathrm{dem}(v) \leq \sup(w)$; and
(c) $C$ induces a connected subgraph of $G$.

Such a set $C \subseteq V$ is called a *supplied set* for $G$. The *fulfillment* $f(C)$ *of a supplied set* $C$ is the sum of demands of all demand vertices in $C$, that is,

$$f(C) = \sum_{v \in C} \mathrm{dem}(v).$$

A supplied set $C$ is called the *maximum supplied set* for $G$ if $f(C)$ is maximum among all supplied sets for $G$. Then the *maximum partition problem* is to find a maximum supplied set for a given graph $G$. The *maximum fulfillment* $f(G)$ *of a graph* $G$ is the fulfillment $f(C)$ of the maximum supplied set $C$ for $G$. For the series-parallel graph $G$ in Fig. 1(b), the supplied set $C$ shaded in the figure has the maximum fulfillment, and hence $f(G) = f(C) = 23$, while $f(S) = 25$ for the star $S$ in Fig. 1(c).

**[Main ideas]**

Let $G$ be a series-parallel graph, let $u$, $u'$ and $u''$ be nodes of a binary decomposition tree $T$ of $G$, and let $G_u = (V_u, E_u)$, $G_{u'} = (V_{u'}, E_{u'})$ and $G_{u''} = (V_{u''}, E_{u''})$ be the subgraphs of $G$ for nodes $u$, $u'$ and $u''$, respectively, as illustrated in

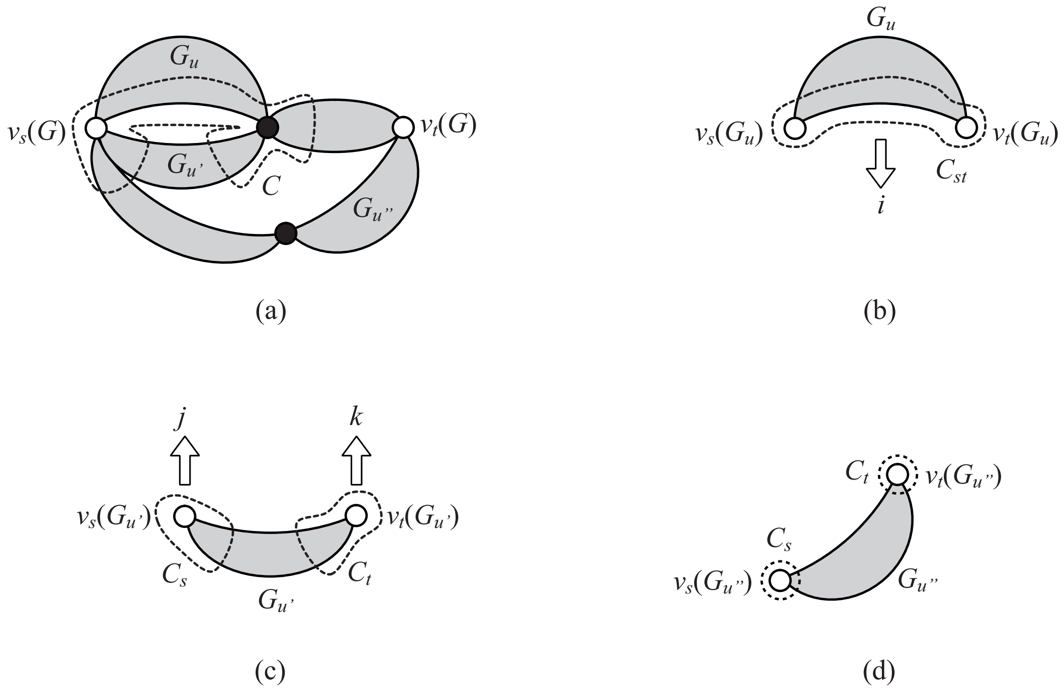



Fig. 5. (a) A supplied set $C$ for a series-parallel graph $G$, (b) a connected set $C_{st}$ for $G_u$, (c) a separated pair $(C_s, C_t)$ of sets for $G_{u'}$, and (d) a separated pair $(C_s, C_t)$ of isolated sets for $G_{u''}$.

Fig. 5(a). Every supplied set $C$ for $G$ naturally induces subsets of $V_u$, $V_{u'}$ and $V_{u''}$. The supplied set $C$ for $G$ indicated by a dotted closed curve in Fig. 5(a) induces a single subset $C_{st}$ of $V_u$ in Fig. 5(b) such that $C_{st} = C \cap V_u$ and $v_s(G_u), v_t(G_u) \in C_{st}$. On the other hand, $C$ induces a pair of subsets $C_s$ and $C_t$ of $V_{u'}$ in Fig. 5(c) such that $C_s \cup C_t = C \cap V_{u'}$, $C_s \cap C_t = \emptyset$, $v_s(G_{u'}) \in C_s$ and $v_t(G_{u'}) \in C_t$. A set $C_{st}$, $C_s$ or $C_t$ is not always a supplied set for $G_u$ or $G_{u'}$, because it may not contain the supply vertex $w$. $C_{st}$ is a "connected set" for $G_u$, that is, $C_{st}$ induces a connected subgraph of $G_u$, while the pair $(C_s, C_t)$ is a "separated pair of sets" for $G_{u'}$, that is, $C_s$ and $C_t$ induce vertex-disjoint connected subgraphs of $G_{u'}$. The set $C$ in Fig. 5(a) contains no terminals of $G_{u''}$. In such a case, we regard that $\text{dem}(v_s(G_{u''})) = \text{dem}(v_t(G_{u''})) = 0$ and $C$ induces a separated pair of singleton sets $(C_s, C_t)$ such that $C_s = \{v_s(G_{u''})\}$ and $C_t = \{v_t(G_{u''})\}$, as illustrated in Fig. 5(d). (The formal definitions will be given later.)

If a set $C_{st}$, $C_s$ or $C_t$ contains the supply vertex $w$, then the set may have the "marginal" power, the amount of which is no greater than $\text{sup}(w)$. If a set does not contain $w$, then the set may have the "deficient" power, the amount of which should be no greater than $\text{sup}(w)$. Thus we later introduce five functions $g$, $h_1$, $h_2$, $h_3$ and $h_4$; for a series-parallel graph $G_u$ and a real number $x$, the value $g(G_u, x)$ represents the maximum marginal power or the minimum deficient power of connected sets for $G_u$; for a series-parallel graph $G_u$ and a real number $x$, the value $h_i(G_u, x)$, $1 \leq i \leq 4$, represents the maximum marginal power or the minimum deficient power of separated pairs of sets for $G_u$. Our idea is to compute $g(G_u, x)$ and $h_i(G_u, x)$, $1 \leq i \leq 4$, from the leaves of $T$ to the root $r$ of $T$ by means of dynamic programming.

[**Formal definitions of "connected sets" and "separated pair of sets"**]

We now formally define the notion of connected sets and separated pair of sets for a series-parallel graph $G$. Let $G_u = (V_u, E_u)$ be a subgraph of $G$ for a node $u$ of a binary decomposition tree $T$ of $G$, and let $v_s = v_s(G_u)$ and $v_t = v_t(G_u)$. We call a set $C \subseteq V_u$ a *connected set for* $G_u$ if $C$ satisfies the following three conditions (see Fig. 5(b)):

(a) $v_s, v_t \in C$;
(b) $C$ induces a connected subgraph of $G_u$; and
(c) $\sum_{v \in C} \mathrm{dem}(v) \leq \sup(w)$.

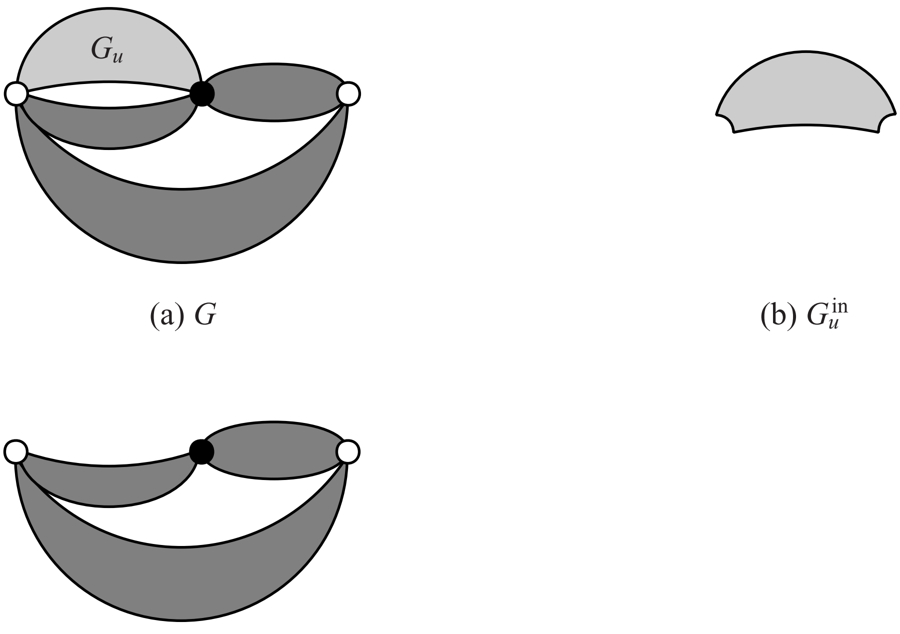
A pair of sets $C_s, C_t \subseteq V_u$ is called a *separated pair* (*of sets*) *for* $G_u$ if $C_s$ and $C_t$ satisfy the following three conditions (see Fig. 5(c)):

(a) $C_s \cap C_t = \emptyset$, $v_s \in C_s$ and $v_t \in C_t$;
(b) $C_s$ and $C_t$ induce connected subgraphs of $G_u$; and
(c) $\sum_{v \in C_s \cup C_t} \mathrm{dem}(v) \leq \sup(w)$.

We then classify connected sets and separated pairs further into smaller classes. The "power flow" around a terminal depends on whether the terminal is a supply vertex or a demand vertex. Since we want to deal with the two cases uniformly, we introduce a virtual graph $G_u^*$ for a subgraph $G_u$ of $G$; $G_u^*$ is obtained from $G_u$ by regarding each of the two terminals $v_s$ and $v_t$ as a demand vertex whose demand is zero. We denote by $\mathrm{dem}^*(x)$ the demand of a demand vertex $x$ in $G_u^*$, and hence

$$\mathrm{dem}^*(x) = \begin{cases} 0 & \text{if } x \text{ is } v_s \text{ or } v_t; \\ \mathrm{dem}(x) & \text{otherwise.} \end{cases}$$

Clearly every connected set for $G_u$ is a connected set for $G_u^*$. However, a connected set $C$ for $G_u^*$ is not necessarily a connected set for $G_u$; for example, if $\sum_{x \in C} \mathrm{dem}^*(x) \leq \sup(w)$ but $\sum_{x \in C} \mathrm{dem}(x) = \mathrm{dem}(v_s) + \mathrm{dem}(v_t) + \sum_{x \in C} \mathrm{dem}^*(x) > \sup(w)$, then $C$ is not a connected set for $G_u$. Similarly, every separated pair for $G_u$ is a separated pair for $G_u^*$, while not every separated pair for $G_u^*$ is a separated pair for $G_u$. We denote by $G_u^{\mathrm{in}}$ the graph obtained from $G_u$ by deleting the two terminals $v_s$ and $v_t$ as illustrated in Fig. 6(b), while we denote by $G_u^{\mathrm{out}}$ the graph obtained from $G$ by deleting all the vertices of $G_u$ except $v_s$ and $v_t$ as illustrated in Fig. 6(c).




(a) $G$ (b) $G_u^{\text{in}}$ (c) $G_u^{\text{out}}$

Fig. 6. (a) A series-parallel graph $G$, (b) a subgraph $G_u^{\text{in}}$ of $G_u$, and (c) a subgraph $G_u^{\text{out}}$ of $G$.

Let $\mathbb{R}_w = \{x \in \mathbb{R} : |x| \le \sup(w)\}$, where $\mathbb{R}$ denotes the set of all real numbers. For each real number $i \in \mathbb{R}_w$, we call a connected set $C$ for $G_u^*$ an $i$-*connected set* if $C$ satisfies the following two conditions (a) and (b):

(a) if $i > 0$, then $w \in C$ and

$$i + \sum_{x \in C} \text{dem}^*(x) \le \sup(w);$$

and

(b) if $i \le 0$, then $w \notin C$ and

$$\sum_{x \in C} \text{dem}^*(x) \le |i| = -i.$$

An $i$-connected set $C$ for $G_u^*$ with $i > 0$ is a supplied set for $G_u^*$, and hence corresponds to some supplied set $C_r$ for the whole graph $G = G_r$ such that $w \in C \subseteq C_r$, where $r$ is the root of $T$; an amount $i$ of the remaining power of $w$ can be delivered outside $G_u$ through $v_s$ or $v_t$; and hence the "margin" of $C$ is $i$. On the other hand, an $i$-connected set $C$ for $G_u^*$ with $i \le 0$ is not a supplied set for $G_u^*$, but may correspond to a supplied set $C_r$ for $G = G_r$ such that $w \notin C \subset C_r$ and $w \in C_r$; an amount $|i|$ of power must be delivered to $C$ from $w$ through $v_s$ or $v_t$, and hence the "deficiency" of $C$ is $|i|$. For an

$i$-connected set $C$ for $G_u^*$, let

$$f(C,i) = \sum_{x \in C} \text{dem}^*(x).$$

Then $f(C,i) = f(C)$ for $G_u^*$ if $0 < i \leq \text{sup}(w)$. On the other hand, if $-\text{sup}(w) \leq i \leq 0$, then $f(C,i)$ represents the fulfillment of $C$ when an amount $|i|$ of power is delivered to $C$ from $w$ in $G_u^{\text{out}}$. According to the definition of an $i$-connected set, a connected set $C$ for $G_u^*$ is not a 0-connected set for $G_u^*$ if $C$ contains the supply vertex $w$ $(\neq v_s, v_t)$ and

$$\sum_{x \in C} \text{dem}^*(x) = \text{sup}(w).$$

Because the demands of $v_s$ and $v_t$ are positive, we have

$$\sum_{x \in C} \text{dem}(x) > \text{sup}(w)$$

and hence such a connected set $C$ for $G_u^*$ is not a connected set for $G_u$ and we need not to take $C$ into account. Thus, if $C$ is a 0-connected set for $G_u^*$, then $C = \{v_s, v_t\}$ and $G_u$ has an edge $(v_s, v_t)$.

Let $\sigma \notin \mathbb{R}_w$ be a symbol. For each pair of $j$ and $k$ in $\mathbb{R}_w \cup \{\sigma\}$, we call a separated pair $(C_s, C_t)$ for $G_u^*$ a $(j,k)$-*separated pair* if $(C_s, C_t)$ satisfies the following seven conditions (a)–(g):

(a) if $j \in \mathbb{R}_w$ and $j > 0$, then $w \in C_s$ and

$$j + \sum_{x \in C_s} \text{dem}^*(x) \leq \text{sup}(w);$$

(b) if $j \in \mathbb{R}_w$ and $j \leq 0$, then $w \notin C_s$ and

$$\sum_{x \in C_s} \text{dem}^*(x) \leq -j;$$

(c) if $j = \sigma$, then $C_s = \{v_s\}$;

(d) if $k \in \mathbb{R}_w$ and $k > 0$, then $w \in C_t$ and

$$k + \sum_{x \in C_t} \text{dem}^*(x) \leq \text{sup}(w);$$

(e) if $k \in \mathbb{R}_w$ and $k \leq 0$, then $w \notin C_t$ and

$$\sum_{x \in C_t} \text{dem}^*(x) \leq -k;$$

(f) if $k = \sigma$, then $C_t = \{v_t\}$; and

(g) if $j, k \in \mathbb{R}_w$ and $j + k \le 0$, then $j \le 0$ and $k \le 0$.

Since there is exactly one supply vertex $w$ in $G$, there is no $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ such that $j > 0$ and $k > 0$. A $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $j > 0$ corresponds to a supplied set $C_r$ for the whole graph $G = G_r$ such that $w \in C_s \subseteq C_r$; an amount $j$ of the remaining power of $w$ can be delivered outside $C_s$ through $v_s$, and hence the margin of $C_s$ is $j$. A $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $j \le 0$ may correspond to a supplied set $C_r$ for $G$ such that $C_s \subset C_r$ and either $w \in C_t$ or $w \in C_r - C_s \cup C_t$; an amount $|j|$ of power must be delivered to $C_s$ through $v_s$, and hence the deficiency of $C_s$ is $|j|$. A $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $j = \sigma$ corresponds to a supplied set $C_r$ for $G$ such that $v_s \notin C_r$, that is, $v_s$ is never supplied power. (See Figs. 5(a) and (d).) Clearly $C_s = \{v_s\}$ if $C_s$ is a $(0, k)$-separated pair for $G_u^*$. A $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $k > 0$, $k \le 0$ or $k = \sigma$ corresponds to a supplied set $C_r$ for $G$ similarly as above. For a $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$, let

$$f(C_s, C_t, j, k) = \begin{cases} \sum_{x \in C_s \cup C_t} \mathrm{dem}^*(x) & \text{if } j, k \in \mathbb{R}_w; \\ \sum_{x \in C_s} \mathrm{dem}^*(x) & \text{if } j \in \mathbb{R}_w \text{ and } k = \sigma; \text{ and} \\ \sum_{x \in C_t} \mathrm{dem}^*(x) & \text{if } j = \sigma \text{ and } k \in \mathbb{R}_w. \end{cases}$$

Let

$$f(\{v_s\}, \{v_t\}, \sigma, \sigma) = \max\{f(C_u) \mid C_u \text{ is a supplied set for } G_u^* \text{ such that } v_s, v_t \notin C_u\},$$

and let $f(\{v_s\}, \{v_t\}, \sigma, \sigma) = 0$ if $G_u^*$ has no supplied set $C_u$ such that $v_s, v_t \notin C_u$.

**[Formal definitions of functions $g$ and $h_i$, $1 \le i \le 4$]**

Let $\mathcal{G}$ denote the set of all series-parallel graphs. We now formally define a function $g : (\mathcal{G}, \mathbb{R}) \to \mathbb{R}_w \cup \{-\infty\}$ as follows: for a series-parallel graph $G_u^* \in \mathcal{G}$ and a real number $x \in \mathbb{R}$,

$$g(G_u^*, x) = \max\{i \in \mathbb{R}_w \mid G_u^* \text{ has an } i\text{-connected set } C \text{ such that } f(C, i) \ge x\}. \tag{7}$$

If $G_u^*$ has no $i$-connected set $C$ with $f(C, i) \ge x$ for any number $i \in \mathbb{R}_w$, then let $g(G_u^*, x) = -\infty$. We then formally define a function $h_1 : (\mathcal{G}, \mathbb{R}) \to \mathbb{R}_w \cup \{-\infty\}$ as follows: for a series-parallel graph $G_u^* \in \mathcal{G}$ and a real number $x \in \mathbb{R}$,

$$h_1(G_u^*, x) = \max\{j + k \mid G_u^* \text{ has a } (j,k)\text{-separated pair } (C_s, C_t) \text{ such that } j, k \in \mathbb{R}_w,\ |j+k| \leq \sup(w), \text{ and } f(C_s, C_t, j, k) \geq x\}. \quad (8)$$

If $G_u^*$ has no $(j,k)$-separated pair $(C_s, C_t)$ with $f(C_s, C_t, j, k) \geq x$ for any pair of numbers $j$ and $k$ in $\mathbb{R}_w$, then let $h_1(G_u^*, x) = -\infty$. It should be noted that a $(j,k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $j, k \in \mathbb{R}_w$ corresponds to a supplied set $C_r$ for $G$ such that $C_s \cup C_t \subseteq C_r$, and hence we can simply take the summation of $j$ and $k$ as the marginal power or the deficient power of $C_s \cup C_t$. We next formally define a function $h_2 : (\mathcal{G}, \mathbb{R}) \to \mathbb{R}_w \cup \{-\infty\}$ as follows: for a series-parallel graph $G_u^* \in \mathcal{G}$ and a real number $x \in \mathbb{R}$,

$$h_2(G_u^*, x) = \max\{j \in \mathbb{R}_w \mid G_u^* \text{ has a } (j,\sigma)\text{-separated pair } (C_s, \{v_t\}) \text{ such that } f(C_s, \{v_t\}, j, \sigma) \geq x\}. \quad (9)$$

If $G_u^*$ has no $(j,\sigma)$-separated pair $(C_s, \{v_t\})$ with $f(C_s, \{v_t\}, j, \sigma) \geq x$ for any number $j \in \mathbb{R}_w$, then let $h_2(G_u^*, x) = -\infty$. We then formally define a function $h_3 : (\mathcal{G}, \mathbb{R}) \to \mathbb{R}_w \cup \{-\infty\}$ as follows: for a series-parallel graph $G_u^* \in \mathcal{G}$ and a real number $x \in \mathbb{R}$,

$$h_3(G_u^*, x) = \max\{k \in \mathbb{R}_w \mid G_u^* \text{ has a } (\sigma,k)\text{-separated pair } (\{v_s\}, C_t) \text{ such that } f(\{v_s\}, C_t, \sigma, k) \geq x\}. \quad (10)$$

If $G_u^*$ has no $(\sigma,k)$-separated pair $(\{v_s\}, C_t)$ with $f(\{v_s\}, C_t, \sigma, k) \geq x$ for any number $k \in \mathbb{R}_w$, then let $h_3(G_u^*, x) = -\infty$. We finally define a function $h_4 : (\mathcal{G}, \mathbb{R}) \to \{0, -\infty\}$ as follows: for a series-parallel graph $G_u^* \in \mathcal{G}$ and a real number $x \in \mathbb{R}$,

$$h_4(G_u^*, x) = \begin{cases} 0 & \text{if } G_u^* \text{ has a } (\sigma,\sigma)\text{-separated pair } (\{v_s\}, \{v_t\}) \\ & \qquad \text{such that } f(\{v_s\}, \{v_t\}, \sigma, \sigma) \geq x; \\ -\infty & \text{otherwise.} \end{cases} \quad (11)$$

Clearly, the five functions $g$ and $h_i$, $1 \leq i \leq 4$, are non-increasing. For any negative real number $x < 0$, we have $g(G_u^*, x) = g(G_u^*, 0)$ and $h_i(G_u^*, x) = h_i(G_u^*, 0)$, $1 \leq i \leq 4$.

Our algorithm computes $g(G_u^*, x)$ and $h_i(G_u^*, x)$, $1 \leq i \leq 4$, for each node $u$ of a binary decomposition tree $T$ of a given series-parallel graph $G$ from the leaves to the root $r$ of $T$ by means of dynamic programming.

We first show how to compute the maximum fulfillment $f(G)$ of a given graph $G$ from $g(G^*, x)$ and $h_i(G^*, x)$, $1 \le i \le 4$.

**[How to compute** $f(G)$**]**

Suppose that $g(G_r^*, x)$ and $h_i(G_r^*, x)$, $1 \le i \le 4$, have been computed for the root $r$ of $T$. Since $G = G_r$, one can easily compute $f(G)$ from $g(G^*, x)$ and $h_i(G^*, x)$, $1 \le i \le 4$, as in the following two cases (a) and (b), where $v_s = v_s(G)$ and $v_t = v_t(G)$.

Case (a): *one of $v_s$ and $v_t$ is the supply vertex $w$ and the other is a demand vertex.*

One may assume without loss of generality that $v_s$ is the supply vertex $w$ and $v_t$ is a demand vertex. Let $C$ be a supplied set for $G$ having the maximum fulfillment. Then there are the following two cases (i) and (ii), as illustrated in Fig. 7:

(i) $v_t$ is supplied power from $v_s$ $(= w)$, that is, $v_s, v_t \in C$; and

(ii) $v_t$ is not supplied power, that is, $v_s \in C$ and $v_t \notin C$.

For Case (i), we compute $f_1(G)$ as follows:

$$f_1(G) = \max\{x + \mathrm{dem}(v_t) \mid x \in \mathbb{R} \text{ and } \mathrm{sup}(w) + g(G^*, x) - \mathrm{dem}(v_t) \ge 0\}. \tag{12}$$

Note that $g(G^*, x) \le 0$ for every number $x \in \mathbb{R}$ since $G^*$ has no supply vertex. If $\mathrm{sup}(w) + g(G^*, x) - \mathrm{dem}(v_t) < 0$ for any number $x \in \mathbb{R}$, then let $f_1(G) = -\infty$.

For Case (ii), we compute $f_2(G)$ as follows:

$$f_2(G) = \max\{x \in \mathbb{R} \mid \mathrm{sup}(w) + h_2(G^*, x) \ge 0\}. \tag{13}$$

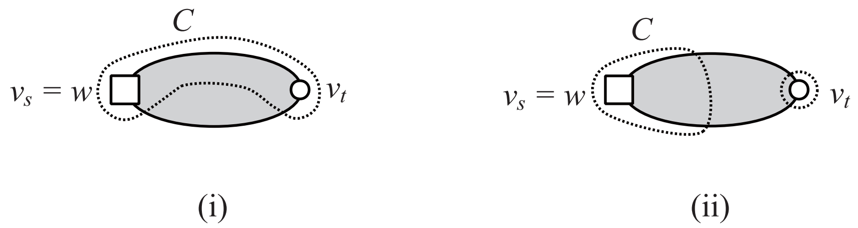



Fig. 7. Two cases in Case (a).

Note that $h_2(G^*, x) \leq 0$ for every number $x \in \mathbb{R}$. If $\sup(w) + h_2(G^*, x) < 0$ for any number $x \in \mathbb{R}$, then let $f_2(G) = -\infty$.

We thus have

$$f(G) = \max\{f_1(G),\ f_2(G)\}. \tag{14}$$

Case (b): *both $v_s$ and $v_t$ are demand vertices.*

Let $C$ be a supplied set for $G$ having the maximum fulfillment. In this case, there are the following four cases (iii)–(vi), as illustrated in Fig. 8:

(iii) $v_s, v_t \in C$;
(iv) $v_s \in C$ and $v_t \notin C$;
(v) $v_s \notin C$ and $v_t \in C$; and
(vi) $v_s, v_t \notin C$.

For Case (iii), we compute $f_3(G)$ as follows:

$$f_3(G) = \max\{x + \text{dem}(v_s) + \text{dem}(v_t) \mid x \in \mathbb{R} \text{ and } g(G^*, x) - \text{dem}(v_s) - \text{dem}(v_t) \geq 0\}. \tag{15}$$

If $g(G^*, x) - \text{dem}(v_s) - \text{dem}(v_t) < 0$ for any number $x \in \mathbb{R}$, then let $f_3(G) = -\infty$.

For Case (iv), we compute $f_4(G)$ as follows:

$$f_4(G) = \max\{x + \text{dem}(v_s) \mid x \in \mathbb{R} \text{ and } h_2(G^*, x) - \text{dem}(v_s) \geq 0\}. \tag{16}$$

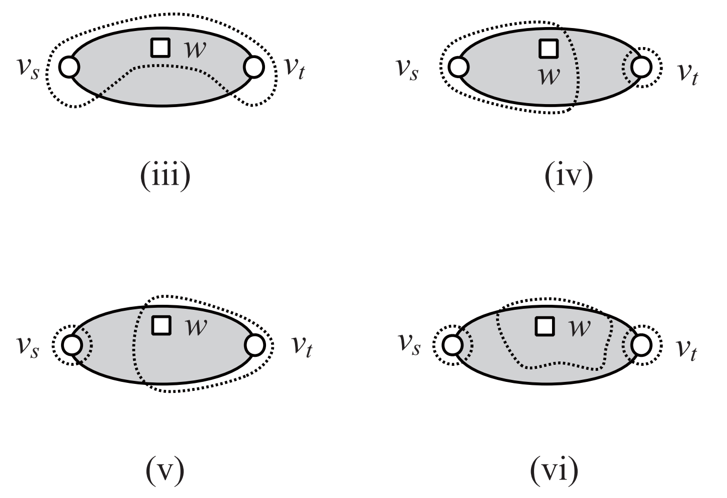



Fig. 8. Four cases in Case (b).

If $h_2(G^*, x) - \mathrm{dem}(v_s) < 0$ for any number $x \in \mathbb{R}$, then let $f_4(G) = -\infty$.

For Case (v), we compute $f_5(G)$ as follows:

$$f_5(G) = \max\{x + \mathrm{dem}(v_t) \mid x \in \mathbb{R} \text{ and } h_3(G^*, x) - \mathrm{dem}(v_t) \geq 0\}. \quad (17)$$

If $h_3(G^*, x) - \mathrm{dem}(v_t) < 0$ for any number $x \in \mathbb{R}$, then let $f_5(G) = -\infty$.

For Case (vi), we compute $f_6(G)$ as follows:

$$f_6(G) = \max\{x \in \mathbb{R} \mid h_4(G^*, x) = 0\}. \quad (18)$$

If $h_4(G^*, x) = -\infty$ for any number $x \in \mathbb{R}$, then let $f_6(G) = -\infty$.

We thus have

$$f(G) = \max\{f_3(G),\ f_4(G),\ f_5(G),\ f_6(G)\}. \quad (19)$$

We then explain how to compute $g(G^*_u, x)$ and $h_i(G^*_u, x)$, $1 \leq i \leq 4$, for each node $u$ of $T$.

**[How to compute $g(G^*_u, x)$ and $h_i(G^*_u, x)$, $1 \leq i \leq 4$]**

We first compute $g(G^*_u, x)$ and $h_i(G^*_u, x)$, $1 \leq i \leq 4$, for each leaf $u$ of $T$, for which $G^*_u$ contains exactly one edge as illustrated in Fig. 3(a). Since the two terminals of $G^*_u$ are demand vertices of demands zero, we have

$$g(G^*_u, x) = \begin{cases} 0 & \text{if } x \leq 0; \\ -\infty & \text{otherwise.} \end{cases} \quad (20)$$

Similarly, for each index $i$, $1 \leq i \leq 4$, we have

$$h_i(G^*_u, x) = \begin{cases} 0 & \text{if } x \leq 0; \\ -\infty & \text{otherwise.} \end{cases} \quad (21)$$

We next compute $g(G^*_u, x)$ and $h_i(G^*_u, x)$, $1 \leq i \leq 4$, for each internal node $u$ of $T$ from the counterparts of the two children of $u$ in $T$. However, we show only how to compute $h_1(G^*_u, x)$ for a p-node $u$ of $T$, because one can similarly

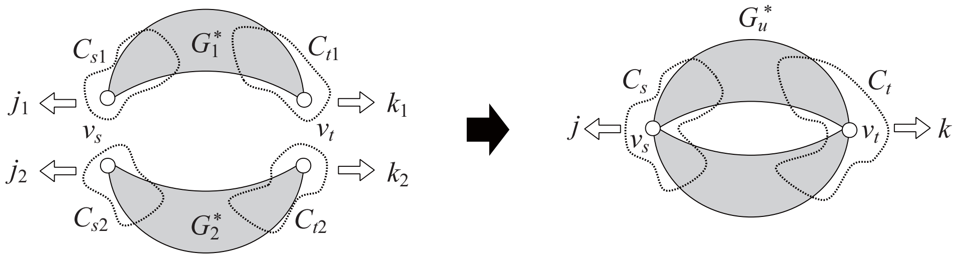



Fig. 9. Combining a $(j_1, k_1)$-separated pair $(C_{s1}, C_{t1})$ for $G_1^*$ and a $(j_2, k_2)$-separated pair $(C_{s2}, C_{t2})$ for $G_2^*$ to a $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^* = G_1^* \parallel G_2^*$ with $j, k \in \mathbb{R}_w$.

compute $g(G_u^*, x)$ and $h_i(G_u^*, x)$, $1 \le i \le 4$, for each p-node and s-node of $T$; the details are given in Appendix A.

We compute $h_1(G_u^*, x)$ for a p-node $u$ of $T$. Let $G_u = G_1 \parallel G_2$, and let $v_s = v_s(G_u^*)$ and $v_t = v_t(G_u^*)$. (See Figs. 3(c) and 9.) Let $(C_s, C_t)$ be a $(j, k)$-separated pair for $G_u^*$ with $j, k \in \mathbb{R}_w$ such that $f(C_s, C_t, j, k) \ge x \in \mathbb{R}$ and $j + k = h_1(G_u^*, x) \ne -\infty$. The $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ can be obtained by combining a $(j_1, k_1)$-separated pair $(C_{s1}, C_{t1})$ for $G_1^*$ with a $(j_2, k_2)$-separated pair $(C_{s2}, C_{t2})$ for $G_2^*$ such that $f(C_s, C_t, j, k) = f(C_{s1}, C_{t1}, j_1, k_1) + f(C_{s2}, C_{t2}, j_2, k_2)$, where $j_1, j_2, k_1, k_2 \in \mathbb{R}_w$ such that $j_1 + j_2 = j$ and $k_1 + k_2 = k$, as illustrated in Fig. 9. Since $f(C_s, C_t, j, k) = f(C_{s1}, C_{t1}, j_1, k_1) + f(C_{s2}, C_{t2}, j_2, k_2) \ge x$, we have $f(C_{s1}, C_{t1}, j_1, k_1) \ge y$ and $f(C_{s2}, C_{t2}, j_2, k_2) \ge x - y$ for some number $y \in \mathbb{R}$. Since $(C_{s1}, C_{t1})$ is a $(j_1, k_1)$-separated pair for $G_1^*$ with $f(C_{s1}, C_{t1}, j_1, k_1) \ge y$, one may assume by Eq. (8) that $j_1 + k_1 = h_1(G_1^*, y)$. Similarly, one may assume that $j_2 + k_2 = h_1(G_2^*, x - y)$. Since $h_1(G_u^*, x) = j + k = (j_1 + j_2) + (k_1 + k_2) = h_1(G_1^*, y) + h_1(G_2^*, x - y)$, one can compute $h_1(G_u^*, x)$ as follows:

$$h_1(G_u^*, x) = \max_y \{h_1(G_1^*, y) + h_1(G_2^*, x - y)\}. \tag{22}$$

It should be noted that the maximum above is taken over all real numbers $y \in \mathbb{R}$ such that

$$\text{if } h_1(G_1^*, y) + h_1(G_2^*, x - y) \le 0 \text{ then } h_1(G_1^*, y) \le 0 \text{ and } h_1(G_2^*, x - y) \le 0.$$

(Remember condition (g) of a $(j, k)$-separated pair.)

### *3.4 Proof of Theorem 2*

We now show that our algorithm takes time $O(F^2 n)$ for a series-parallel graph $G$ as a proof of Theorem 2, where $F$ is an arbitrary upper bound on the maximum fulfillment $f(G)$ of $G$. For example, $F = \min\{\sup(w), \sum_{v \in V} \mathrm{dem}(v)\}$.

Since all demands and the supply in a given series-parallel graph $G$ are integers, $f(C_u)$ is an integer for any supplied set $C_u$ for $G_u$. Similarly, $f(C,i)$ and $f(C_s,C_t,j,k)$ are integers for any $i$-connected set $C$ and any $(j,k)$-separated pair $(C_s,C_t)$ for $G_u^*$, respectively. We denote by $\mathbb{Z}$ the set of all integers. Let $\mathbb{Z}_w = \{x \in \mathbb{Z} : |x| \leq w\}$. Define a function $\hat{g} : (\mathcal{G},\mathbb{Z}) \to \mathbb{Z}_w \cup \{-\infty\}$ similarly as $g : (\mathcal{G},\mathbb{R}) \to \mathbb{R}_w \cup \{-\infty\}$ in Eq. (7): for a series-parallel graph $G_u^* \in \mathcal{G}$ and an integer $x \in \mathbb{Z}$, we define

$$\hat{g}(G_u^*,x) = \max\{i \in \mathbb{Z}_w \mid G_u^* \text{ has an } i\text{-connected set } C \text{ such that } f(C,i) \geq x\}.$$

Define functions $\hat{h}_1, \hat{h}_2, \hat{h}_3 : (\mathcal{G},\mathbb{Z}) \to \mathbb{Z}_w \cup \{-\infty\}$ and $\hat{h}_4 : (\mathcal{G},\mathbb{Z}) \to \{0,-\infty\}$ similarly as $h_1, h_2, h_3$ and $h_4$ in Eqs. (8)–(11). Define integral values $\hat{f}_i(G)$, $1 \leq i \leq 6$, similarly as $f_i(G)$, $1 \leq i \leq 6$, in Eqs. (12), (13) and (15)–(18), respectively. Then clearly $\hat{f}_i(G) = f_i(G)$, $1 \leq i \leq 6$, since all demands and the supply in $G$ are integers. Therefore, by Eqs. (14) and (19) we can compute $f(G)$ from $\hat{f}_i(G)$, $1 \leq i \leq 6$. We shall thus compute values $\hat{g}(G_u^*,x)$ and $\hat{h}_i(G_u^*,x)$, $1 \leq i \leq 4$, for all integers $x \in \mathbb{Z}$. However, one can easily observe that it suffices to compute them only for integers $x \in \mathbb{Z}_F^+$, where $\mathbb{Z}_F^+ = \{x \in \mathbb{Z} \mid 0 \leq x \leq F\}$; remember that $F$ is an upper bound of the maximum fulfillment $f(G)$ of $G$.

For each leaf $u$ of $T$ and all integers $x \in \mathbb{Z}_F^+$, one can easily compute values $\hat{g}(G_u^*,x)$ and $\hat{h}_i(G_u^*,x)$, $1 \leq i \leq 4$, in time $O(|\mathbb{Z}_F^+|) = O(F)$ by the counterparts of Eqs. (20) and (21). Since $G$ is a series-parallel simple graph of $n$ vertices, $G$ has at most $2n-3$ edges and hence $T$ has at most $2n-3$ leaves. One can thus compute $\hat{g}(G_u^*,x)$ and $\hat{h}_i(G_u^*,x)$, $1 \leq i \leq 4$, for all leaves $u$ of $T$ in time $O(Fn)$.

For each internal node $u$ of $T$ and all integers $x \in \mathbb{Z}_F^+$, one can compute $\hat{g}(G_u^*,x)$ and $\hat{h}_i(G_u^*,x)$, $1 \leq i \leq 4$, in time $O(|\mathbb{Z}_F^+|^2) = O(F^2)$ by the counterparts of Eq. (22) in Subsection 3.3 and Eqs. (A.1)–(A.16) in Appendix A. Since $T$ has at most $2n-4$ internal nodes, one can compute $\hat{g}(G^*,x)$ and $\hat{h}_i(G^*,x)$, $1 \leq i \leq 4$, in time $O(F^2 n)$.

One can compute the maximum fulfillment $f(G)$ of $G$ from $\hat{g}(G^*,x)$ and $\hat{h}_i(G^*,x)$, $1 \leq i \leq 4$, in time $O(F)$ by the counterparts of Eqs. (12)–(19).

Thus the maximum partition problem can be solved in time $O(F^2 n)$. This completes a proof of Theorem 2. $\square$

## 4 FPTAS

Assume in this section that the supply and all demands are positive real numbers which are not always integers. Since the maximum partition problem is MAXSNP-hard, there is no PTAS for the problem on general graphs unless P = NP. However, using the pseudo-polynomial-time algorithm in Section 3, we can obtain an FPTAS for series-parallel graphs having exactly one supply vertex, and have the following theorem.

**Theorem 3** *There is a fully polynomial-time approximation scheme for the maximum partition problem on a series-parallel graph having exactly one supply vertex.*

In the remainder of this section, as a proof of Theorem 3, we give an algorithm to find a supplied set $C$ for a series-parallel graph $G$ with $f(C) \geq (1-\varepsilon)f(G)$ in time polynomial in $n$ and $1/\varepsilon$ for any real number $\varepsilon$, $0 < \varepsilon < 1$, where $n$ is the number of vertices in $G$. Thus our approximate maximum fulfillent $\bar{f}(G)$ of $G$ is $f(C)$, and hence the error is bounded by $\varepsilon f(G)$, that is,

$$f(G) - \bar{f}(G) = f(G) - f(C) \leq \varepsilon f(G). \tag{23}$$

We now outline our algorithm and the analysis. We extend the ordinary "scaling and rounding" technique for the knapsack problem [5,9] and the maximum partition problem on trees [7] and apply it to the maximum partition problem for a series-parallel graph with a single supply vertex. For some scaling factor $t$, we consider the set $\{\cdots, -2t, -t, 0, t, 2t, \cdots\}$ as the range of functions $g$ and $h_i$, $1 \leq i \leq 4$, and find the approximate solution $\bar{f}(G)$ by using the pseudo-polynomial-time algorithm in Section 3. As we will show later in Lemma 2(b), we have

$$f(G) - \bar{f}(G) < 4nt. \tag{24}$$

Intuitively, Eq. (24) holds because the series and parallel connections are executed no more than $2n$ times and each connection adds at most $2t$ to the error $f(G) - \bar{f}(G)$. Choosing an appropriate upper bound $F$ such that $F/2 \leq f(G) \leq F$, and taking $t = \varepsilon F/(8n)$, we have Eq. (23).

One may expect that an FPTAS could be obtained simply by using an ordinary scaling and rounding technique and the pseudo-polynomial-time algorithm, as follows:

(1) scale down the supply $\mathrm{sup}(w)$ by $\overline{\mathrm{sup}}(w) = \lfloor \mathrm{sup}(w)/t \rfloor$, and scale up the demand $\mathrm{dem}(v)$ by $\overline{\mathrm{dem}}(v) = \lceil \mathrm{dem}(v)/t \rceil$ for each demand vertex $v$,

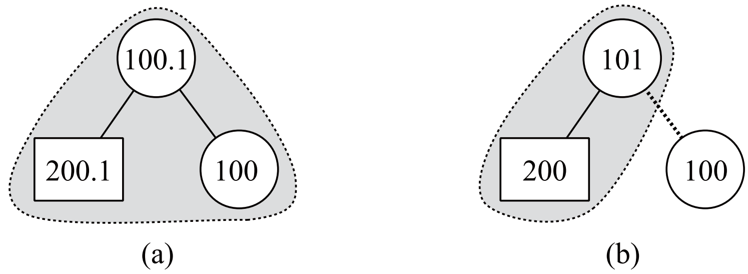



Fig. 10. (a) Original problem instance, and (b) instance scaled by factor $t = 1$.

(2) find a supplied set $C$ for $G$ having the maximum fulfillment for the scaled instance by using the pseudo-polynomial-time algorithm,
(3) compute the fulfillment $f(C)$ for the original instance, and
(4) output $f(C)$ as an approximate maximum fulfillment $\bar{f}(G)$ for the original one.

Although such a straightforward method always finds a feasible solution for the original instance, the error $f(G) - \bar{f}(G)$ cannot be bounded by $4nt$. Consider an example in Fig. 10, where the supply vertex is drawn by a rectangle and each demand vertex by a circle. Figure 10(a) depicts an original instance, while Fig. 10(b) depicts an instance scaled by factor $t = 1$. For the original instance, the supplied set shaded in Fig. 10(a) has the maximum fulfillment $f(G) = 200.1$. On the other hand, for the scaled one, the supplied set $C$ shaded in Fig. 10(b) is found by the pseudo-polynomial-time algorithm, and $C$ has a fulfillment of $f(C) = \bar{f}(G) = 100.1$ for the original one. Thus $f(G) - \bar{f}(G) = 100$, and hence $f(G) - \bar{f}(G)$ cannot be bounded by $4nt = 12$. Similarly, one can easily observe that $f(G) - \bar{f}(G)$ cannot be bounded by $cnt$ for any fixed constant $c$. Thus the straightforward method above cannot yield an FPTAS.

We now give the details of our algorithm and the proof of its correctness. For a positive real number $t$, let $\mathbb{R}^t = \{\cdots, -2t, -t, 0, t, 2t, \cdots\}$ and $\mathbb{R}_F^{t+} = \{x \in \mathbb{R}^t \mid 0 \le x \le F\}$. The functions $g$ and $h_i$, $1 \le i \le 4$, in Section 3 have range $\mathbb{R}$. In this section we define new functions $\bar{g}$, $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$ and $\bar{h}_4$ which have a sampled range $\mathbb{R}^t$ and approximate $g$, $h_1$, $h_2$, $h_3$ and $h_4$, respectively. It should be noted that $\bar{g}$ and $\bar{h}_i$, $1 \le i \le 4$, do not always take the same value as $g$ and $h_i$, $1 \le i \le 4$, respectively, even for $x \in \mathbb{R}^t$. More precisely, we

(i) define $\bar{g}$ and $\bar{h}_i$, $1 \le i \le 4$, for $x \in \mathbb{R}^t$ by the counterparts of Eqs. (7)–(11), and recursively compute $\bar{g}$ and $\bar{h}_i$, $1 \le i \le 4$, for $x \in \mathbb{R}^t$ by the counterparts of Eqs. (20)–(22) and Eqs. (A.1)–(A.16) in Appendix A;
(ii) define and compute values $\bar{f}_i(G)$, $1 \le i \le 6$, by the counterparts of Eqs. (12), (13) and (15)–(18); and
(iii) define and compute $\bar{f}(G)$ as follows:

$$\bar{f}(G) = \max\{\bar{f}_1(G),\ \bar{f}_2(G)\} \tag{25}$$

if one of $v_s(G)$ and $v_t(G)$ is the supply vertex $w$, and

$$\bar{f}(G) = \max\{\bar{f}_3(G),\ \bar{f}_4(G),\ \bar{f}_5(G),\ \bar{f}_6(G)\} \tag{26}$$

if both $v_s(G)$ and $v_t(G)$ are demand vertices.

We will show later in Lemma 2(b) that $\bar{f}(G)$ is an approximate value of $f(G)$ satisfying Eq. (24). It should be noted that the demands and the supply are never scaled and rounded when we compute the functions $\bar{g}$ and $\bar{h}_i$, $1 \le i \le 4$, as above, and hence these functions take real values which are not necessarily in $\mathbb{R}^t$.

Let $T$ be a binary decomposition tree of $G$. We denote by $n(T)$ the number of nodes in $T$. For a node $u$ of $T$, we denote by $T_u$ a subtree of $T$ which is rooted at $u$ and is induced by all descendants of $u$ in $T$. We denote by $n(T_u)$ the number of nodes in $T_u$. The functions $\bar{g}$ and $\bar{h}_i$, $1 \le i \le 4$, approximate the original functions $g$ and $h_i$, $1 \le i \le 4$, as in the following lemma. Note that $\bar{g}(G^*_u, x) = \bar{g}(G^*_u, 0)$ and $\bar{h}_i(G^*_u, x) = \bar{h}_i(G^*_u, 0)$, $1 \le i \le 4$, for any negative number $x \in \mathbb{R}^t$.

**Lemma 1** *For each node $u$ of a binary decomposition tree $T$ of $G$, the following* (a) *and* (b) *hold*:

(a) (i) $\bar{g}(G^*_u, x) \le g(G^*_u, x)$ *for any number* $x \in \mathbb{R}^t$;

(ii) $\bar{g}(G^*_u, x)$ *is non-increasing*; *and*

(iii) *for any number $x \in \mathbb{R}$, there is an integer $\alpha$ such that*

$$0 \le \alpha \le n(T_u) - 1$$

*and*

$$\bar{g}(G^*_u, \lfloor x/t \rfloor t - \alpha t) \ge g(G^*_u, x),$$

*and*

(b) *for each index $i$, $1 \le i \le 4$,*

(i) $\bar{h}_i(G^*_u, x) \le h_i(G^*_u, x)$ *for any number* $x \in \mathbb{R}^t$;

(ii) $\bar{h}_i(G^*_u, x)$ *is non-increasing*; *and*

(iii) *for any number $x \in \mathbb{R}$, there is an integer $\beta_i$ such that*

$$0 \le \beta_i \le n(T_u) - 1$$

*and*

$$\bar{h}_i(G^*_u, \lfloor x/t \rfloor t - \beta_i t) \ge h_i(G^*_u, x).$$

*Proof.* See Appendix B. □

We then have the following lemma.

**Lemma 2** *The following* (a) *and* (b) *hold*:
(a) *for each index* $i$, $1 \le i \le 6$,

$$f_i(G) - n(T)t \le \bar{f}_i(G);$$

*and*
(b) $f(G) - 4nt < \bar{f}(G) \le f(G)$.

*Proof.* (a) We prove only for the index $i = 1$, that is,

$$f_1(G) - n(T)t \le \bar{f}_1(G), \tag{27}$$

because one can similarly prove for the other indices.

Let $v_s = v_s(G)$ and $v_t = v_t(G)$. One may assume that $v_s = w$ for $f_1(G)$ and $\bar{f}_1(G)$. Let $x$ be a real number such that

$$x + \mathrm{dem}(v_t) = f_1(G) \ne -\infty, \tag{28}$$

then by Eq. (12) we have

$$\sup(w) + g(G^*, x) - \mathrm{dem}(v_t) \ge 0. \tag{29}$$

By Lemma 1(a) there is an integer $\alpha$ such that

$$0 \le \alpha \le n(T) - 1 \tag{30}$$

and

$$\bar{g}(G^*, \lfloor x/t \rfloor t - \alpha t) \ge g(G^*, x). \tag{31}$$

By Eqs. (29) and (31) we have

$$\sup(w) + \bar{g}(G^*, \lfloor x/t \rfloor t - \alpha t) - \mathrm{dem}(v_t) \ge 0.$$

Therefore, by the counterpart of Eq. (12) we have

$$\bar{f}_1(G) \ge \lfloor x/t \rfloor t - \alpha t + \mathrm{dem}(v_t). \tag{32}$$

By Eqs. (28), (30) and (32) we have

$$\begin{aligned}\bar{f}_1(G) &\geq \lfloor x/t \rfloor\, t - (n(T)-1)t + \mathrm{dem}(v_t)\\ &= (\lfloor x/t \rfloor t + t) - n(T)t + \mathrm{dem}(v_t)\\ &\geq x - n(T)t + \mathrm{dem}(v_t)\\ &\geq f_1(G) - n(T)t.\end{aligned}$$

We have thus verified Eq. (27).

(b) By Lemma 1(a), Eq. (12) and its counterpart, we have $\bar{f}_1(G) \leq f_1(G)$. Similarly we have $\bar{f}_i(G) \leq f_i(G)$, $2 \leq i \leq 6$. Therefore by Eqs. (14), (19), (25) and (26) we have $\bar{f}(G) \leq f(G)$. By Lemma 2(a) and Eqs. (14), (19), (25) and (26) we have

$$f(G) - n(T)t \leq \bar{f}(G).$$

Since $G$ is a series-parallel simple graph, $G$ has at most $2n-3$ edges and hence $T$ has at most $2n-3$ leaves. Therefore $T$ has at most $4n-7$ nodes and hence $n(T) < 4n$. We thus have $f(G) - 4nt < \bar{f}(G)$. □

We are now ready to prove Theorem 3.

**Proof of Theorem 3.**

One may assume without loss of generality that, for each demand vertex $v$ of a finite demand, a series-parallel graph $G$ has a path $Q(v)$ going from the supply vertex $w$ to $v$ such that the sum of demands on $Q(v)$ does not exceed $\sup(w)$, and hence $v$ is contained in some supplied set $C$ for $G$. Otherwise, $v$ cannot be contained in any supplied set, and hence one can regard that $v$ has an infinite demand. (We do not delete such a vertex $v$ from $G$, because the resulting graph may not be series-parallel.) One can examine in polynomial time whether there exists such a path $Q(v)$ for each demand vertex $v$ in $G$; this can be done for all demand vertices $v$ in $G$ in time $O(n^2)$ simply by applying a single-source shortest path algorithm to a graph similar to a "line-graph" of $G$.

One may assume that $G$ has one or more demand vertices of finite demands; otherwise, $f(G) = 0$. Let $V'$ be the set consisting of the supply vertex $w$ and all demand vertices of finite demands in $G$. Let $G'$ be a subgraph of $G$ induced by $V'$, then $G'$ is connected. Let $m_d = \max\{\mathrm{dem}(v) \mid v \in V'\}$, and let $v'$ be a demand vertex such that $\mathrm{dem}(v') = m_d$. Then $G'$ has a supplied set $C$ containing $v'$, and $C$ is a supplied set also for $G$. We thus have

$$f(G) \geq f(C) \geq \mathrm{dem}(v') = m_d. \tag{33}$$

We now choose an upper bound $F$ on $f(G)$ such that

$$\frac{F}{2} \leq f(G) \leq F. \tag{34}$$

Consider a simple greedy algorithm to find a supplied set for $G'$. The algorithm traverses $G'$ by the breadth-first search starting from $w$, and includes traversed demand vertices in a supplied set as much as possible so that the set induces a connected subgraph of $G'$ and the sum of demands in the set does not exceed $\sup(w)$. Let $C_A$ be a supplied set for $G'$ found by the greedy algorithm. If either $f(C_A) = \sup(w)$ or $f(C_A) = \sum_{v \in V'} \mathrm{dem}(v)$, then $C_A$ is the maximum supplied set for $G'$ and hence for $G$. One may thus assume without loss of generality that $f(C_A) < \sup(w)$ and $f(C_A) < \sum_{v \in V'} \mathrm{dem}(v)$. Then, there are demand vertices in $G'$ which were traversed but could not be included in $C_A$. Let $v''$ be the vertex, among these vertices, that was first traversed. Then we have

$$\sup(w) < f(C_A) + \mathrm{dem}(v''). \tag{35}$$

We choose $F$ as follows:

$$F = 2 \cdot \max\{f(C_A), m_d\}. \tag{36}$$

Then, since $f(G) \leq \sup(w)$ and $\mathrm{dem}(v'') \leq m_d$, by Eqs. (35) and (36) we have

$$\begin{aligned} f(G) &< f(C_A) + \mathrm{dem}(v'') \\ &\leq f(C_A) + m_d \\ &\leq 2 \cdot \max\{f(C_A), m_d\} \\ &= F. \end{aligned}$$

Since $C_A$ is a supplied set for $G$, we have $f(G) \geq f(C_A)$ and hence by Eqs. (33) and (36)

$$f(G) \geq \max\{f(C_A), m_d\} = \frac{F}{2}.$$

We have thus verified Eq. (34).

Let

$$t = \frac{\varepsilon F}{8n}. \tag{37}$$

Then by Lemma 2(b) and Eqs. (34) and (37) we have

$$f(G) < \bar{f}(G) + 4n\frac{\varepsilon F}{8n} \leq \bar{f}(G) + \varepsilon f(G),$$

and hence we have Eq. (23).

One can observe that the algorithm takes time

$$O\left(\left|\mathbb{R}_F^{t+}\right|^2 n\right) = O\left(\frac{n^3}{\varepsilon^2}\right),$$

because $|\mathbb{R}_F^{t+}| = \lfloor F/t \rfloor + 1$, and hence by Eq. (37) we have $F/t \leq 8n/\varepsilon$. □

## 5 Conclusions

In this paper, we studied the approximability of the maximum partition problem. We first showed that the maximum partition problem is MAXSNP-hard. We then gave an FPTAS for series-parallel graphs having exactly one supply vertex. It is easy to modify the FPTAS so that it actually finds a supplied set for a series-parallel graph. The FPTAS for series-parallel graphs can be extended to that for partial $k$-trees although it would become much more complicated.

In the ordinary knapsack problem, each "item" is assigned a "size" and "value," and one wishes to choose a subset of items that maximizes the sum of values of items such that their total size does not exceed the size of a bag [5,9]. Consider a slightly modified version of the maximum partition problem on graphs in which each demand vertex is assigned not only a demand but also a "value," and one wishes to find a partition which maximizes the sum of values of all demand vertices in components with supply vertices. This problem is indeed a generalization of the ordinary knapsack problem, and can be solved for series-parallel graphs and partial $k$-trees using techniques similar to those for the maximum partition problem if there is exactly one supply vertex. Note that the standard approximation methods for the knapsack problem in [5,9] cannot be applied to the modified maximum partition problem.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## A How to compute $g(G_u^*, x)$ and $h_i(G_u^*, x)$, $1 \leq i \leq 4$

In this section, we explain how to compute $g(G_u^*, x)$ and $h_i(G_u^*, x)$, $1 \leq i \leq 4$, for each internal node $u$ of $T$ from the counterparts of the two children of $u$ in $T$.

We first consider a parallel connection.

**[Parallel connection]**

Let $G_u = G_1 \parallel G_2$, and let $v_s = v_s(G_u^*)$ and $v_t = v_t(G_u^*)$. (See Figs. 3(c) and A.1–A.3.)

We have shown in Section 3 that one can compute $h_1(G_u^*, x)$ in Eq. (8) by Eq. (22).

We now show how to compute $h_2(G_u^*, x)$ in Eq. (9). For $j \in \mathbb{R}_w$, every $(j, \sigma)$-separated pair $(C_s, \{v_t\})$ for $G_u^*$ with $f(C_s, \{v_t\}, j, \sigma) \geq x$ can be obtained by combining a $(j_1, \sigma)$-separated pair $(C_{s1}, \{v_t\})$ for $G_1^*$ with a $(j_2, \sigma)$-separated pair $(C_{s2}, \{v_t\})$ for $G_2^*$ such that $j_1, j_2 \in \mathbb{R}_w$, $j_1 + j_2 = j$ and $f(C_s, \{v_t\}, j, \sigma) = f(C_{s1}, \{v_t\}, j_1, \sigma) + f(C_{s2}, \{v_t\}, j_2, \sigma)$, as illustrated in Fig. A.1. We can thus compute $h_2(G_u^*, x)$ as follows:

$$h_2(G_u^*, x) = \max_y \{h_2(G_1^*, y) + h_2(G_2^*, x - y)\} \tag{A.1}$$

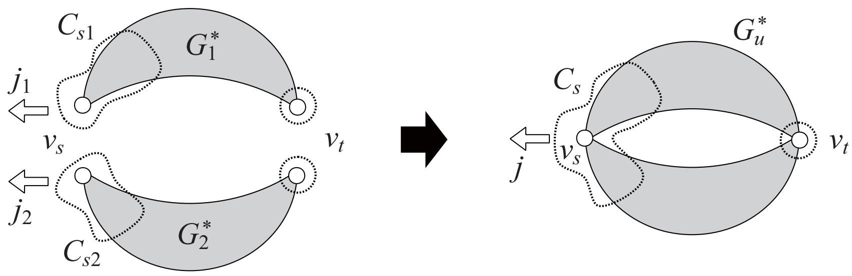


Fig. A.1. Combining a $(j_1, \sigma)$-separated pair $(C_{s1}, \{v_t\})$ for $G_1^*$ and a $(j_2, \sigma)$-separated pair $(C_{s2}, \{v_t\})$ for $G_2^*$ to a $(j, \sigma)$-separated pair $(C_s, \{v_t\})$ for $G_u^* = G_1^* \parallel G_2^*$.

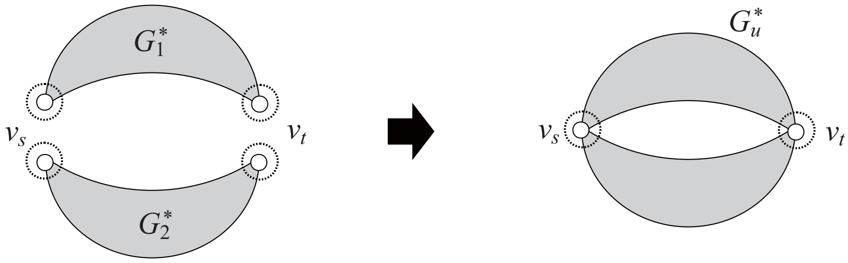


Fig. A.2. Combining a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_1^*$ and a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_2^*$ to a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_u^* = G_1^* \parallel G_2^*$.

where the maximum above is taken over all real numbers $y \in \mathbb{R}$ such that

if $h_2(G_1^*, y) + h_2(G_2^*, x-y) \le 0$
then $h_2(G_1^*, y) \le 0$ and $h_2(G_2^*, x-y) \le 0$.

One can compute $h_3(G_u^*, x)$ in Eq. (10) similarly as $h_2(G_u^*, x)$.

We then show how to compute $h_4(G_u^*, x)$ in Eq. (11). Every $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_u^*$ with $f(\{v_s\},\{v_t\},\sigma,\sigma) \ge x$ can be obtained by combining a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_1^*$ with a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_2^*$ such that $f(\{v_s\},\{v_t\},\sigma,\sigma) = f(\{v_s\},\{v_t\},\sigma,\sigma) + f(\{v_s\},\{v_t\},\sigma,\sigma)$, as illustrated in Fig. A.2. We can thus compute $h_4(G_u^*, x)$ as follows:

$$h_4(G_u^*, x) = \max\{h_4(G_1^*, y) + h_4(G_2^*, x-y) \mid y \in \mathbb{R}\}. \tag{A.2}$$

We next show how to compute $g(G_u^*, x)$ in Eq. (7). There are the following two cases (a) and (b) where an $i$-connected set $C$ for $G_u^*$ with $f(C,i) \ge x$ is formed from the counterparts of $u$'s children, as illustrated in Figs. A.3(a) and (b). We define two functions $g^a$ and $g^b$ for the two cases (a) and (b), respectively.

Case (a): *$C$ is obtained by combining an $i_1$-connected set $C_1$ for $G_1^*$ with an $i_2$-connected set $C_2$ for $G_2^*$ such that $f(C,i) = f(C_1,i_1) + f(C_2,i_2)$ and $i_1 + i_2 = i$.* (See Fig. A.3(a).)

We define $g^a(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$g^a(G_u^*, x) = \max_y \{g(G_1^*, y) + g(G_2^*, x-y)\} \tag{A.3}$$

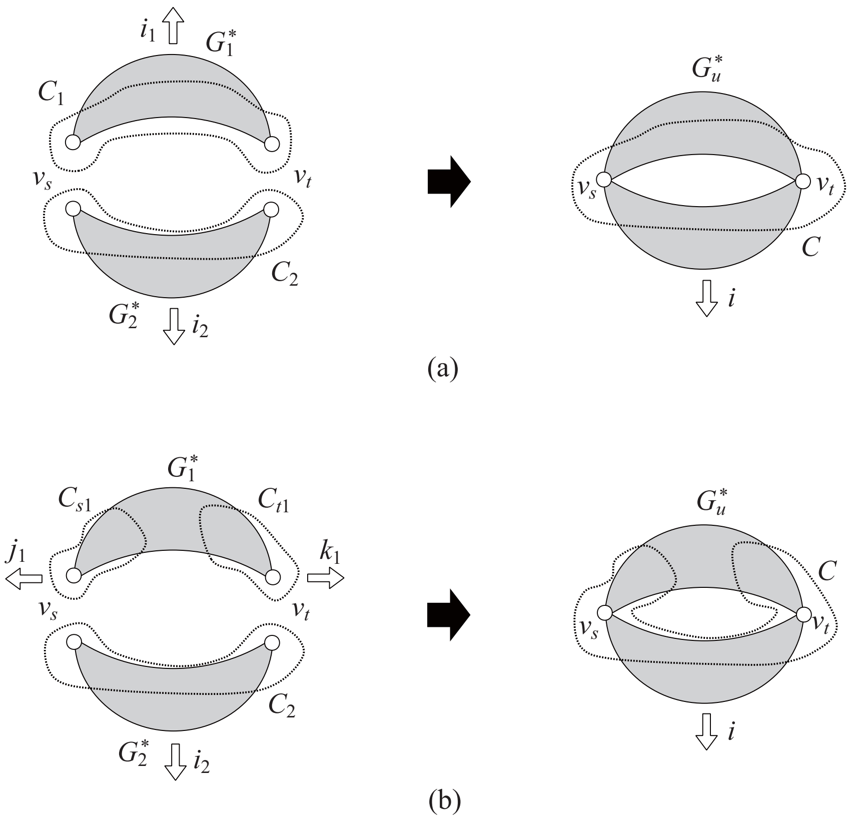


Fig. A.3. Forming an $i$-connected set $C$ for $G^*_u = G^*_1 \parallel G^*_2$.

where the maximum above is taken over all real numbers $y \in \mathbb{R}$ such that

$$\text{if } g(G^*_1, y) + g(G^*_2, x - y) \leq 0 \text{ then } g(G^*_1, y) \leq 0 \text{ and } g(G^*_2, x - y) \leq 0.$$

Case (b): *$C$ is obtained by either combining a connected set for $G^*_1$ with a separated pair for $G^*_2$ or combining a separated pair for $G^*_1$ with a connected set for $G^*_2$.*

One may assume without loss of generality that an $i$-connected set $C$ is obtained by combining a $(j_1, k_1)$-separated pair $(C_{s1}, C_{t1})$ for $G^*_1$ with an $i_2$-connected set $C_2$ for $G^*_2$ such that $f(C, i) = f(C_{s1}, C_{t1}, j_1, k_1) + f(C_2, i_2)$, where $j_1 + k_1 + i_2 = i$. (See Fig. A.3(b).)

We define $g^b(G^*_u, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$g^b(G^*_u, x) = \max_y \{h_1(G^*_1, y) + g(G^*_2, x - y)\} \tag{A.4}$$

where the maximum above is taken over all real numbers $y \in \mathbb{R}$ such that

$$\text{if } h_1(G^*_1, y) + g(G^*_2, x - y) \leq 0 \text{ then } h_1(G^*_1, y) \leq 0 \text{ and } g(G^*_2, x - y) \leq 0.$$

From $g^a$ and $g^b$ above, one can compute $g(G_u^*, x)$ as follows:

$$g(G_u^*, x) = \max\{g^a(G_u^*, x),\ g^b(G_u^*, x)\}. \tag{A.5}$$

We next consider a series connection.

**[Series connection]**

Let $G_u = G_1 \bullet G_2$, and let $v$ be the vertex of $G$ identified by the series connection, that is, $v = v_t(G_1) = v_s(G_2)$. (See Figs. 3(b) and A.4–A.7.) We define $sd(v)$ as follows:

$$sd(v) = \begin{cases} \sup(v) & \text{if } v \text{ is a supply vertex,} \\ -\text{dem}(v) & \text{if } v \text{ is a demand vertex.} \end{cases}$$

Remember that $\text{dem}(w) = 0$ for the supply vertex $w$.

We first show how to compute $g(G_u^*, i)$ in Eq. (7). For $i \in \mathbb{R}_w$, every $i$-connected set $C$ for $G_u^*$ with $f(C, i) \geq x$ can be obtained by combining an $i_1$-connected set $C_1$ for $G_1^*$ with an $i_2$-connected set $C_2$ for $G_2^*$ such that $f(C, i) = f(C_1, i_1) + f(C_2, i_2) + \text{dem}(v)$ and $i_1 + i_2 + sd(v) = i$, as illustrated in Fig. A.4. Therefore $g(G_u^*, x)$ can be computed for each real number $x \in \mathbb{R}$, as follows:

$$g(G_u^*, x) = \max_{y_1, y_2}\{g(G_1^*, y_1) + g(G_2^*, y_2) + sd(v)\} \tag{A.6}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that

(a) $y_1, y_2 \in \mathbb{R}$;
(b) $y_1 + y_2 + \text{dem}(v) = x$; and
(c) if $g(G_1^*, y_1) + g(G_2^*, y_2) + sd(v) \leq 0$, then $g(G_1^*, y_1) \leq 0$, $g(G_2^*, y_2) \leq 0$ and $sd(v) < 0$.

We next show how to compute $h_1(G_u^*, x)$ in Eq. (8). There are the following two cases (a) and (b) where a $(j, k)$-separated pair $(C_s, C_t)$ for $G_u^*$ with $f(C_s, C_t, j, k) \geq x$ is formed from the counterparts of $u$'s children, as illustrated in Figs. A.5(a) and (b). We define two functions $h_1^a$ and $h_1^b$ for the two cases (a) and (b), respectively.

Case (a): $(C_s, C_t)$ *is obtained by either combining a connected set for* $G_1^*$ *with a separated pair for* $G_2^*$ *or combining a separated pair for* $G_1^*$ *with a connected set for* $G_2^*$.

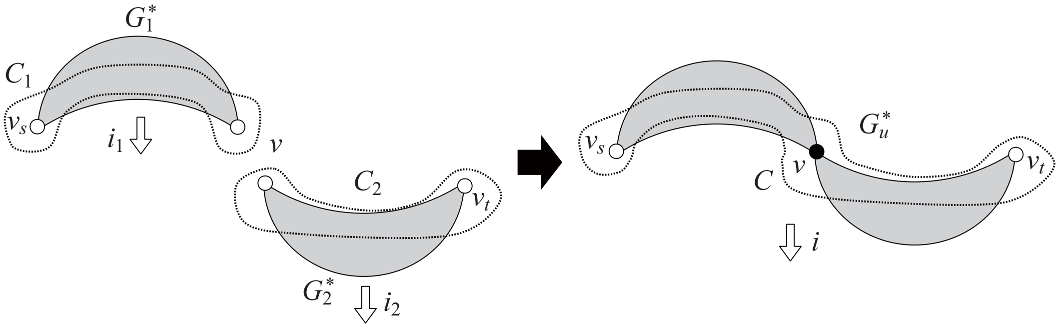


Fig. A.4. Combining an $i_1$-connected set $C_1$ for $G_1^*$ and an $i_2$-connected set $C_2$ for $G_2^*$ to an $i$-connected set $C$ for $G_u^*$, where $G_u = G_1 \bullet G_2$.

One may assume without loss of generality that a $(j,k)$-separated pair $(C_s, C_t)$ is obtained by combining an $i_1$-connected set $C_1$ for $G_1^*$ with a $(j_2,k)$-separated pair $(C_{s2}, C_t)$ for $G_2^*$ such that $f(C_s, C_t, j, k) = f(C_1, i_1) + f(C_{s2}, C_t, j_2, k) + \mathrm{dem}(v)$ and $i_1 + j_2 + sd(v) = i$. (See Fig. A.5(a).)

We define $h_1^a(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$h_1^a(G_u^*, x) = \max_{y_1, y_2}\{g(G_1^*, y_1) + h_1(G_2^*, y_2) + sd(v)\} \tag{A.7}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that
  (a) $y_1, y_2 \in \mathbb{R}$;

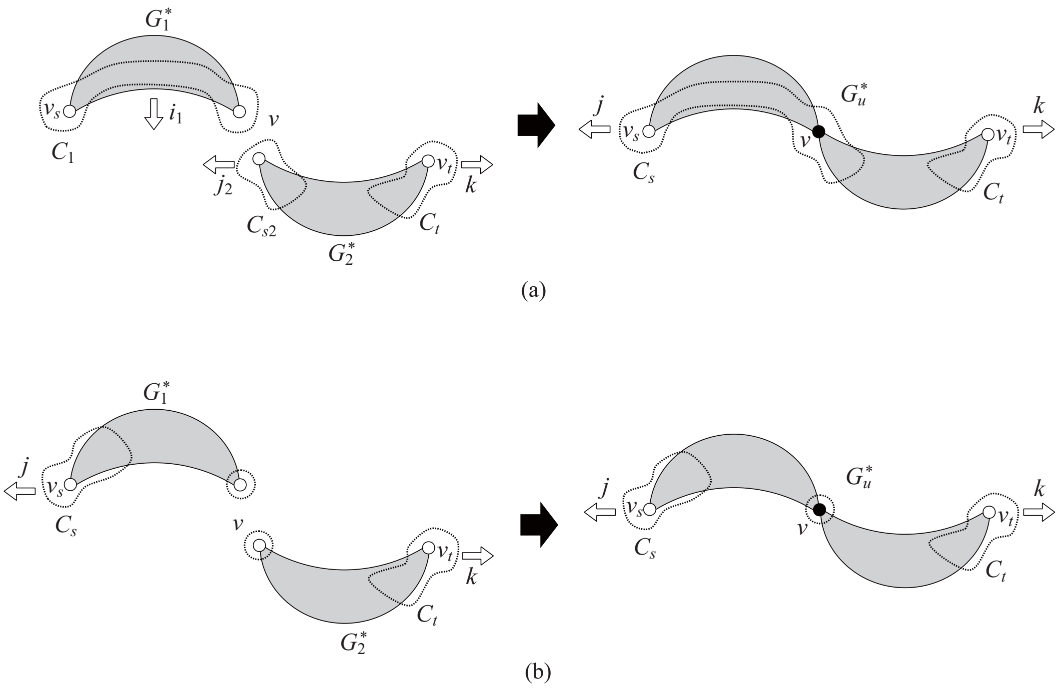


Fig. A.5. Forming a $(j,k)$-separated pair $(C_s, C_t)$ for $G_u^*$, where $j, k \in \mathbb{R}_w$ and $G_u = G_1 \bullet G_2$.

(b) $y_1 + y_2 + \mathrm{dem}(v) = x$; and
(c) if $g(G_1^*, y_1) + h_1(G_2^*, y_2) + sd(v) \leq 0$, then $g(G_1^*, y_1) \leq 0$, $h_1(G_2^*, y_2) \leq 0$ and $sd(v) < 0$.

Case (b): $(C_s, C_t)$ *is obtained by combining a* $(j, \sigma)$*-separated pair* $(C_s, \{v\})$ *for* $G_1^*$ *with a* $(\sigma, k)$*-separated pair* $(\{v\}, C_t)$ *for* $G_2^*$ *such that* $f(C_s, C_t, j, k) = f(C_s, \{v\}, j, \sigma) + f(\{v\}, C_t, \sigma, k)$. (See Fig. A.5(b).)

We define $h_1^b(G_u^*, x)$ for each real number $x \in \mathbb{R}$ as follows:

$$h_1^b(G_u^*, x) = \max_{y_1, y_2}\{h_2(G_1^*, y_1) + h_3(G_2^*, y_2)\} \tag{A.8}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that
(a) $y_1, y_2 \in \mathbb{R}$;
(b) $y_1 + y_2 = x$; and
(c) if $h_2(G_1^*, y_1) + h_3(G_2^*, y_2) \leq 0$, then $h_2(G_1^*, y_1) \leq 0$ and $h_3(G_2^*, y_2) \leq 0$.

If $v$ is the supply vertex $w$, then let $h_1^b(G_u^*, x) = -\infty$ for each real number $x \in \mathbb{R}$; since $G_u^*$ has a supplied set $C = \{v\}$, the (demand) vertices in $C_s \cup C_t$ cannot be supplied power; note that such a case is regarded as a $(\sigma, \sigma)$-separated pair for $G_u^*$.

From $h_1^a$ and $h_1^b$ above, one can compute $h_1(G_u^*, x)$ as follows:

$$h_1(G_u^*, x) = \max\{h_1^a(G_u^*, x),\ h_1^b(G_u^*, x)\}. \tag{A.9}$$

We then show how to compute $h_2(G_u^*, x)$ in Eq. (9). There are the following two cases (a) and (b) where a $(j, \sigma)$-separated pair $(C_s, \{v_t\})$ for $G_u^*$ with $f(C_s, \{v\}, j, \sigma) \geq x$ is formed from the counterparts of $u$'s children, as illustrated in Figs. A.6(a) and (b). We define two functions $h_2^a$ and $h_2^b$ for the two cases (a) and (b), respectively.

Case (a): $(C_s, \{v_t\})$ *is obtained by combining an* $i_1$*-connected set* $C_1$ *for* $G_1^*$ *with a* $(j_2, \sigma)$*-separated pair* $(C_{s2}, \{v_t\})$ *for* $G_2^*$ *such that* $f(C_s, \{v_t\}, j, \sigma) = f(C_1, i_1) + f(C_{s2}, \{v_t\}, j_2, \sigma) + \mathrm{dem}(v)$ *and* $i_1 + j_2 + sd(v) = j$. (See Fig. A.6(a).)

We define $h_2^a(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$h_2^a(G_u^*, x) = \max_{y_1, y_2}\{g(G_1^*, y_1) + h_2(G_2^*, y_2) + sd(v)\} \tag{A.10}$$

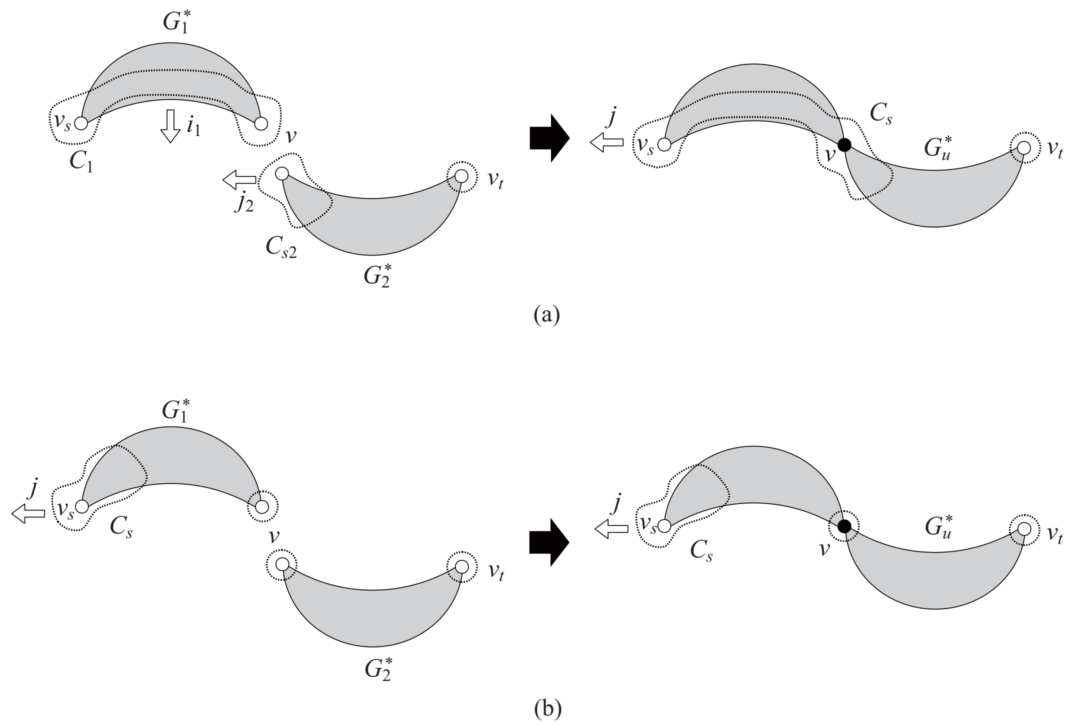


Fig. A.6. Forming a $(j,\sigma)$-separated pair $(C_s,\{v_t\})$ for $G_u^*$, where $j \in \mathbb{R}_w$ and $G_u = G_1 \bullet G_2$.

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that

(a) $y_1, y_2 \in \mathbb{R}$;
(b) $y_1 + y_2 + \mathrm{dem}(v) = x$; and
(c) if $g(G_1^*, y_1) + h_2(G_2^*, y_2) + sd(v) \le 0$, then $g(G_1^*, y_1) \le 0$, $h_2(G_2^*, y_2) \le 0$ and $sd(v) < 0$.

Case (b): $(C_s,\{v_t\})$ *is obtained by combining a* $(j,\sigma)$*-separated pair* $(C_s,\{v\})$ *for* $G_1^*$ *with a* $(\sigma,\sigma)$*-separated pair* $(\{v\},\{v_t\})$ *for* $G_2^*$ *such that* $f(C_s,\{v_t\},j,\sigma) = f(C_s,\{v\},j,\sigma) + f(\{v\},\{v_t\},\sigma,\sigma)$. (See Fig. A.6(b).)

We define $h_2^b(G_u^*, x)$ for each real number $x \in \mathbb{R}$ as follows:

$$h_2^b(G_u^*, x) = \max_{y_1,y_2}\{h_2(G_1^*, y_1) + h_4(G_2^*, y_2)\} \tag{A.11}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that

(a) $y_1, y_2 \in \mathbb{R}$;
(b) $y_1 + y_2 = x$; and
(c) if $h_2(G_1^*, y_1) + h_4(G_2^*, y_2) \le 0$, then $h_2(G_1^*, y_1) \le 0$ and $h_4(G_2^*, y_2) \le 0$.

If $v$ is the supply vertex $w$, then let $h_2^b(G_u^*, x) = -\infty$ for each real number $x \in \mathbb{R}$.

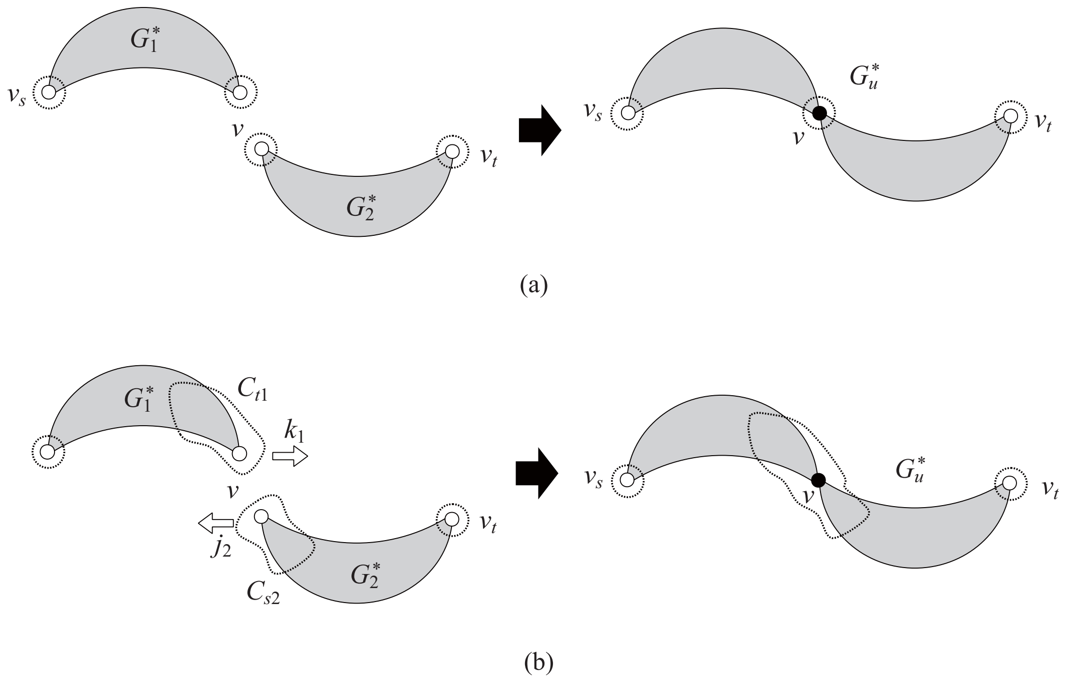


Fig. A.7. Forming a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_u^*$, where $G_u = G_1 \bullet G_2$.

From $h_2^a$ and $h_2^b$ above, one can compute $h_2(G_u^*, x)$ as follows:

$$h_2(G_u^*, x) = \max\{h_2^a(G_u^*, x),\ h_2^b(G_u^*, x)\}. \tag{A.12}$$

One can compute $h_3(G_u^*, x)$ in Eq. (10) similarly as $h_2(G_u^*, x)$.

We finally show how to compute $h_4(G_u^*, x)$ in Eq. (11). There are the following two cases (a) and (b) where a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v_t\})$ for $G_u^*$ with $f(\{v_s\},\{v_t\},\sigma,\sigma) \geq x$ is formed from the counterparts of $u$'s children, as illustrated in Figs. A.7(a) and (b). We define two functions $h_4^a$ and $h_4^b$ for the two cases (a) and (b), respectively.

Case (a): *$(\{v_s\},\{v_t\})$ is obtained by combining a $(\sigma,\sigma)$-separated pair $(\{v_s\},\{v\})$ for $G_1^*$ with a $(\sigma,\sigma)$-separated pair $(\{v\},\{v_t\})$ for $G_2^*$ such that $f(\{v_s\},\{v_t\},\sigma,\sigma)$ $= f(\{v_s\},\{v\},\sigma,\sigma) + f(\{v\},\{v_t\},\sigma,\sigma)$.* (See Fig. A.7(a).)

We define $h_4^a(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$h_4^a(G_u^*, x) = \max_{y_1, y_2}\{h_4(G_1^*, y_1) + h_4(G_2^*, y_2)\} \tag{A.13}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that

$y_1 + y_2 = x$.

Case (b): $(\{v_s\}, \{v_t\})$ *is obtained by combining a* $(\sigma, k_1)$*-separated pair* $(\{v_s\}, C_{t1})$ *for* $G_1^*$ *with a* $(j_2, \sigma)$*-separated pair* $(C_{s2}, \{v_t\})$ *for* $G_2^*$ *such that* $f(\{v_s\}, \{v_t\}, \sigma, \sigma) = f(\{v_s\}, C_{t1}, \sigma, k_1) + f(C_{s2}, \{v_t\}, j_2, \sigma) + \mathrm{dem}(v)$. (See Fig. A.7(b).)

We first define $h'_4(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$h'_4(G_u^*, x) = \max_{y_1, y_2}\{h_3(G_1^*, y_1) + h_2(G_2^*, y_2) + sd(v)\} \tag{A.14}$$

where the maximum above is taken over all real numbers $y_1$ and $y_2$ such that $y_1 + y_2 + \mathrm{dem}(v) = x$. Then $h'_4(G_u^*, x) \geq 0$ if $C_{t1} \cup C_{s2}$ is a supplied set for $G_u^*$, otherwise, $h'_4(G_u^*, x) < 0$. Thus, we define $h_4^b(G_u^*, x)$ for each real number $x \in \mathbb{R}$, as follows:

$$h_4^b(G_u^*, x) = \begin{cases} 0 & \text{if } h'_4(G_u^*, x) \geq 0; \\ -\infty & \text{otherwise.} \end{cases} \tag{A.15}$$

From $h_4^a$ and $h_4^b$ above, one can compute $h_4(G_u^*, x)$ as follows:

$$h_4(G_u^*, x) = \max\{h_4^a(G_u^*, x),\ h_4^b(G_u^*, x)\}. \tag{A.16}$$

## B Proof of Lemma 1

We inductively prove Lemmas 1(a) and (b).

**1° Proof for each subgraph $G_u^*$ corresponding to a leaf $u$ of $T$.**

We prove only that (a) holds for $G_u^*$, because one can similarly prove that (b) holds for $G_u^*$. Since $G_u^*$ contains exactly one edge and the two terminals of $G_u^*$ are demand vertices of demands zero, by Eq. (20) and its counterpart we have

$$\bar{g}(G_u^*, x) = g(G_u^*, x) = \begin{cases} 0 & \text{if } x \leq 0; \\ -\infty & \text{otherwise} \end{cases} \tag{B.1}$$

for any number $x \in \mathbb{R}^t$, and hence (a)(i) holds.

Equation (B.1) implies that (a)(ii) holds for $G_u^*$.

We finally prove that (a)(iii) holds for $G_u^*$. By Eqs. (20) and (B.1) we have

$$\bar{g}\left(G_u^*, \left\lfloor \frac{x}{t} \right\rfloor t\right) \geq g(G_u^*, x)$$

for any number $x \in \mathbb{R}$, and hence (a)(iii) holds for $\alpha = 0 = n(T_u) - 1$.

$2^\circ$ **Induction hypothesis**

Let $u$ be an internal node of $T$, and let $G_1$ and $G_2$ be subgraphs of $G_u$ corresponding to the two children of $u$ in $T$. Let $T_1$ and $T_2$ be subtrees of $T$ rooted at the two children of $u$ in $T$. Suppose that (a) and (b) hold for $G_1$ and $G_2$.

$3^\circ$ **Proof for each subgraph $G_u^*$ corresponding to an internal node $u$ of $T$.**

We first consider a parallel connection, that is, $G_u = G_1 \parallel G_2$. We prove only that (a) holds for $G_u^*$, because one can similarly prove that (b) holds for $G_u^*$.

We first prove that (a)(i) holds for $G_u^*$. Let $x$ be any number in $\mathbb{R}^t$. By the counterpart of Eq. (A.5) we have

$$\bar{g}(G_u^*, x) = \max\{\bar{g}^a(G_u^*, x),\ \bar{g}^b(G_u^*, x)\}.$$

We assume that

$$\bar{g}(G_u^*, x) = \bar{g}^a(G_u^*, x). \tag{B.2}$$

(Proofs for the other cases are similar.) Then by the counterpart of Eq. (A.3) there is a number $y \in \mathbb{R}^t$ such that

$$\bar{g}^a(G_u^*, x) = \bar{g}(G_1^*, y) + \bar{g}(G_2^*, x - y). \tag{B.3}$$

By the induction hypothesis, (a)(i) holds for $G_1^*$ and $G_2^*$, and hence we have

$$\bar{g}(G_1^*, y) \leq g(G_1^*, y) \tag{B.4}$$

and

$$\bar{g}(G_2^*, x - y) \leq g(G_2^*, x - y). \tag{B.5}$$

Substituting Eqs. (B.4) and (B.5) to Eq. (B.3), we have

$$\bar{g}^a(G_u^*, x) \leq g(G_1^*, y) + g(G_2^*, x - y). \tag{B.6}$$

By Eqs. (A.3), (A.5), (B.2) and (B.6) we have

$$\begin{aligned}\bar{g}(G_u^*, x) &= \bar{g}^a(G_u^*, x) \\ &\leq g(G_1^*, y) + g(G_2^*, x - y) \\ &\leq g^a(G_u^*, x) \\ &\leq g(G_u^*, x).\end{aligned}$$

We have thus proved that (a)(i) holds for $G_u^*$.

We then prove that (a)(ii) holds for $G_u^*$. It suffices to prove that

$$\bar{g}(G_u^*, x + t) \leq \bar{g}(G_u^*, x)$$

for any number $x \in \mathbb{R}^t$. By the counterpart of Eq. (A.5) we have

$$\bar{g}(G_u^*, x + t) = \max\{\bar{g}^a(G_u^*, x + t),\ \bar{g}^b(G_u^*, x + t)\}.$$

We assume that

$$\bar{g}(G_u^*, x + t) = \bar{g}^a(G_u^*, x + t). \tag{B.7}$$

(Proofs for the other cases are similar.) Then by the counterpart of Eq. (A.3) there is a number $y \in \mathbb{R}^t$ such that

$$\bar{g}^a(G_u^*, x + t) = \bar{g}(G_1^*, y) + \bar{g}(G_2^*, x - y + t). \tag{B.8}$$

By the induction hypothesis, (a)(ii) holds for $G_2^*$, and hence we have

$$\bar{g}(G_2^*, x - y + t) \leq \bar{g}(G_2^*, x - y). \tag{B.9}$$

Substituting Eq. (B.9) to Eq. (B.8), we have

$$\bar{g}^a(G_u^*, x + t) \leq \bar{g}(G_1^*, y) + \bar{g}(G_2^*, x - y). \tag{B.10}$$

By Eqs. (B.7) and (B.10) and the counterparts of Eqs. (A.3) and (A.5) we have

$$\begin{aligned}\bar{g}(G_u^*, x + t) &= \bar{g}^a(G_u^*, x + t) \\ &\leq \bar{g}(G_1^*, y) + \bar{g}(G_2^*, x - y)\end{aligned}$$

$$\le \bar{g}^a(G_u^*, x)$$
$$\le \bar{g}(G_u^*, x).$$

We have thus proved that (a)(ii) holds for $G_u^*$.

We finally prove that (a)(iii) holds for $G_u^*$. Let $x$ be any real number in $\mathbb{R}$. By Eq. (A.5) we have

$$g(G_u^*, x) = \max\{g^a(G_u^*, x),\ g^b(G_u^*, x)\}.$$

We assume that

$$g(G_u^*, x) = g^a(G_u^*, x). \tag{B.11}$$

(Proofs for the other cases are similar.) Then by Eq. (A.3) there is a real number $y \in \mathbb{R}$ such that

$$g^a(G_u^*, x) = g(G_1^*, y) + g(G_2^*, x - y). \tag{B.12}$$

By the induction hypothesis, (a)(iii) holds for $G_1^*$ and $G_2^*$, and hence there are two integers $\alpha'$ and $\alpha''$ such that

$$0 \le \alpha' \le n(T_1) - 1, \tag{B.13}$$
$$0 \le \alpha'' \le n(T_2) - 1, \tag{B.14}$$
$$g(G_1^*, y) \le \bar{g}\left(G_1^*, \left\lfloor \frac{y}{t} \right\rfloor t - \alpha' t\right) \tag{B.15}$$

and

$$g(G_2^*, x - y) \le \bar{g}\left(G_2^*, \left\lfloor \frac{x - y}{t} \right\rfloor t - \alpha'' t\right). \tag{B.16}$$

Substituting Eqs. (B.15) and (B.16) to Eq. (B.12), we have

$$g^a(G_u^*, x) \le \bar{g}\left(G_1^*, \left\lfloor \frac{y}{t} \right\rfloor t - \alpha' t\right) + \bar{g}\left(G_2^*, \left\lfloor \frac{x - y}{t} \right\rfloor t - \alpha'' t\right). \tag{B.17}$$

By Eq. (B.17) and the counterparts of Eqs. (A.3) and (A.5) we have

$$g^a(G_u^*, x) \le \bar{g}^a\left(G_u^*, \left\lfloor \frac{y}{t} \right\rfloor t + \left\lfloor \frac{x - y}{t} \right\rfloor t - \alpha' t - \alpha'' t\right)$$
$$\le \bar{g}\left(G_u^*, \left\lfloor \frac{y}{t} \right\rfloor t + \left\lfloor \frac{x - y}{t} \right\rfloor t - \alpha' t - \alpha'' t\right). \tag{B.18}$$

Since

$$\left\lfloor \frac{y}{t} \right\rfloor + \left\lfloor \frac{x-y}{t} \right\rfloor > \left\lfloor \frac{x}{t} \right\rfloor - 2,$$

by (a)(ii) we have

$$\begin{aligned} &\bar{g}\left(G_u^*, \left\lfloor \frac{y}{t} \right\rfloor t + \left\lfloor \frac{x-y}{t} \right\rfloor t - \alpha' t - \alpha'' t\right) \\ &\qquad \le \bar{g}\left(G_u^*, \left\lfloor \frac{x}{t} \right\rfloor t - (\alpha' + \alpha'' + 2)t\right). \end{aligned} \tag{B.19}$$

Therefore, by Eqs. (B.11), (B.18) and (B.19) we have

$$\begin{aligned} g(G_u^*, x) &= g^a(G_u^*, x) \\ &\le \bar{g}\left(G_u^*, \left\lfloor \frac{y}{t} \right\rfloor t + \left\lfloor \frac{x-y}{t} \right\rfloor t - \alpha' t - \alpha'' t\right) \\ &\le \bar{g}\left(G_u^*, \left\lfloor \frac{x}{t} \right\rfloor t - (\alpha' + \alpha'' + 2)t\right). \end{aligned}$$

Let $\alpha = \alpha' + \alpha'' + 2 > 0$. Then by Eqs. (B.13) and (B.14) we have

$$\begin{aligned} \alpha &= \alpha' + \alpha'' + 2 \\ &\le (n(T_1) - 1) + (n(T_2) - 1) + 2 \\ &= n(T_1) + n(T_2) \\ &= n(T_u) - 1. \end{aligned}$$

We have thus proved that (a)(iii) holds for $G_u^*$.

We next consider a series connection, that is, $G_u = G_1 \bullet G_2$. We prove only that (a) holds for $G_u^*$, because one can similarly prove that (b) holds for $G_u^*$.

We first prove that (a)(i) holds for $G_u^*$. Let $x$ be any number in $\mathbb{R}^t$. By the counterpart of Eq. (A.6), there are two real numbers $y_1, y_2 \in \mathbb{R}^t$ such that

$$\bar{g}(G_u^*, x) = \bar{g}(G_1^*, y_1) + \bar{g}(G_2^*, y_2) + sd(v) \tag{B.20}$$

and

$$y_1 + y_2 + \mathrm{dem}(v) = x. \tag{B.21}$$

By the induction hypothesis, (a)(i) holds for $G_1^*$ and $G_2^*$, and hence we have

$$\bar{g}(G_1^*, y_1) \le g(G_1^*, y_1) \tag{B.22}$$

and

$$\bar{g}(G_2^*, y_2) \leq g(G_2^*, y_2). \tag{B.23}$$

Substituting Eqs. (B.22) and (B.23) to Eq. (B.20) and using Eqs. (A.6) and (B.21), we have

$$\begin{aligned}\bar{g}(G_u^*, x) &\leq g(G_1^*, y_1) + g(G_2^*, y_2) + sd(v) \\ &\leq g(G_u^*, x).\end{aligned}$$

We have thus proved that (a)(i) holds for $G_u^*$.

We then prove that (a)(ii) holds for $G_u^*$. It suffices to prove that

$$\bar{g}(G_u^*, x + t) \leq \bar{g}(G_u^*, x)$$

for any real number $x \in \mathbb{R}^t$. By the counterpart of Eq. (A.6) there are two real numbers $y_1, y_2 \in \mathbb{R}^t$ such that

$$\bar{g}(G_u^*, x + t) = \bar{g}(G_1^*, y_1) + \bar{g}(G_2^*, y_2 + t) + sd(v) \tag{B.24}$$

and

$$y_1 + y_2 + \mathrm{dem}(v) = x. \tag{B.25}$$

By the induction hypothesis, (a)(ii) holds for $G_2^*$, and hence we have

$$\bar{g}(G_2^*, y_2 + t) \leq g(G_2^*, y_2). \tag{B.26}$$

Substituting Eq. (B.26) to Eq. (B.24) and using Eq. (B.25) and the counterpart of Eq. (A.6), we have

$$\begin{aligned}\bar{g}(G_u^*, x + t) &\leq \bar{g}(G_1^*, y_1) + \bar{g}(G_2^*, y_2) + sd(v) \\ &\leq \bar{g}(G_u^*, x).\end{aligned}$$

We have thus proved that (a)(ii) holds for $G_u^*$.

We finally prove that (a)(iii) holds for $G_u^*$. Let $x$ be any real number in $\mathbb{R}$. By Eq. (A.6) there are two real numbers $y_1, y_2 \in \mathbb{R}$ such that

$$g(G_u^*, x) = g(G_1^*, y_1) + g(G_2^*, y_2) + sd(v) \tag{B.27}$$

and

$$y_1 + y_2 + \mathrm{dem}(v) = x. \tag{B.28}$$

By the induction hypothesis, (a)(iii) holds for $G_1^*$ and $G_2^*$, and hence there are two integers $\alpha'$ and $\alpha''$ such that

$$0 \le \alpha' \le n(T_1) - 1, \tag{B.29}$$
$$0 \le \alpha'' \le n(T_2) - 1, \tag{B.30}$$
$$g(G_1^*, y_1) \le \bar{g}\left(G_1^*, \left\lfloor \frac{y_1}{t} \right\rfloor t - \alpha' t\right) \tag{B.31}$$

and

$$g(G_2^*, y_2) \le \bar{g}\left(G_2^*, \left\lfloor \frac{y_2}{t} \right\rfloor t - \alpha'' t\right). \tag{B.32}$$

Substituting Eqs. (B.31) and (B.32) to Eq. (B.27), we have

$$g(G_u^*, x) \le \bar{g}\left(G_1^*, \left\lfloor \frac{y_1}{t} \right\rfloor t - \alpha' t\right) + \bar{g}\left(G_2^*, \left\lfloor \frac{y_2}{t} \right\rfloor t - \alpha'' t\right) + sd(v). \tag{B.33}$$

By Eq. (B.33) and the counterpart of Eq. (A.6) we have

$$g(G_u^*, x) \le \bar{g}\left(G_u^*, \left\lfloor \frac{y_1}{t} \right\rfloor t + \left\lfloor \frac{y_2}{t} \right\rfloor t + \mathrm{dem}(v) - \alpha' t - \alpha'' t\right). \tag{B.34}$$

By Eq. (B.28) we have

$$\left\lfloor \frac{y_1}{t} \right\rfloor t + \left\lfloor \frac{y_2}{t} \right\rfloor t + \mathrm{dem}(v) \ge \left\lfloor \frac{x}{t} \right\rfloor t - 2t,$$

and hence by Eq. (B.34) and (a)(ii) we have

$$g(G_u^*, x) \le \bar{g}\left(G_u^*, \left\lfloor \frac{x}{t} \right\rfloor t - (\alpha' + \alpha'' + 2)t\right). \tag{B.35}$$

Let $\alpha = \alpha' + \alpha'' + 2 > 0$. Then by Eqs. (B.29) and (B.30) we have

$$\begin{aligned}\alpha &= \alpha' + \alpha'' + 2\\ &\le (n(T_1) - 1) + (n(T_2) - 1) + 2\\ &= n(T_1) + n(T_2)\\ &= n(T_u) - 1.\end{aligned}$$

We have thus proved that (a)(iii) holds for $G_u^*$. $\square$