# Symmetric Assembly Puzzles are Hard, Beyond a Few Pieces

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The goal of a 2D *assembly puzzle* is to arrange a given set of polygonal pieces in a way that they do not overlap and form a target polygonal silhouette. For example, there are over 5,000 Tangram assembly puzzles [4], and many more similar 2D assembly puzzles; see, e.g., [2]. A recent trend in the puzzle world is a relatively new type of 2D assembly puzzle which we call *symmetric assembly puzzles*. In these puzzles the target shape is not specified. Instead, the objective is to rearrange the polygonal pieces so that they form a *symmetric* silhouette (as before, overlap of the pieces is not allowed).

The first symmetric assembly puzzle, "Symmetrix", was designed in 2003 by Japanese puzzle designer Tadao Kitazawa, and was distributed as his exchange puzzle at the 2004 International Puzzle Party (IPP) in Tokyo [3]. In this paper, we aim for an arrangement that creates mirror symmetry (reflection through a line), but other symmetries such as central symmetry or 180° rotation (or all of the above) could be considered. The lack of a target shape specification makes these puzzles quite difficult to solve in practice, even for relatively few and simple pieces.

In this paper, we study the computational complexity of symmetric assembly puzzles in their general form. Given $n$ simple polygons $P_1, P_2, \ldots, P_n$, with $m_1, m_2, \ldots, m_n$ vertices respectively, the goal is to find a mirror-symmetric polygon that can be exactly covered by $P_1, P_2, \ldots, P_n$. We may either allow or forbid the pieces to flip over (reflect). Given the difficulty humans have with few low-complexity shapes, we consider two different generalizations: bounded piece complexity ($m_i = O(1)$) and bounded piece number ($n = O(1)$). In the former case, we prove strong NP-hardness, while in the latter case, we solve the problem in polynomial time (but the exponent depends on the number of pieces).

**Theorem 1** *Symmetric assembly puzzles are strongly NP-hard even if each piece is a polyomino that has at most six vertices and its area is upper bounded by a polynomial function of the number of pieces.*

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We reduce from the Rectangle Packing Puzzle problem, known to be strongly NP-hard [1]. Specifically, it is (strongly) NP-complete to decide whether $n$ given rectangular pieces—sized $1 \times x_1, 1 \times x_2, \ldots, 1 \times x_n$, where the $x_i$'s are positive integers bounded above by a polynomial in $n$—can be exactly packed into a specified rectangular box of area $x_1 + x_2 + \cdots + x_n$.

Let $I = (x_1, \ldots, x_n, w, h)$ be a rectangle packing puzzle. Without loss of generality, we assume that $w \geq h$. Now let $I' = (P_1, \ldots, P_n, F)$ be the symmetric assembly puzzle where $P_i$ is the $1 \times x_i$ rectangle for each $i \in \{1, \ldots, n\}$, and $F$ is the polyomino in Figure 1. We call $F$ the *frame piece* of $I'$. We show that $I$ has a rectangle packing if and only if $I'$ has a symmetric assembly.

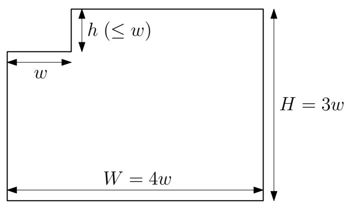


Figure 1: The frame piece $F$.

Clearly, if $I$ has a rectangle packing, then the pieces $P_1, \ldots, P_n$ can be packed into a $w \times h$ rectangle, which combined with the frame piece create a $W \times H$ rectangle. A rectangle is mirror-symmetric, solving the symmetric assembly puzzle. Now we show the reverse implication. Assume that $I'$ has a symmetric arrangement, and let $O^*$ be a mirror-symmetric polygon formed by the pieces $\{P_1, \ldots, P_n, F\}$. We claim that $O^*$ must be a $W \times H$ rectangle, which will imply that $I$ is a yes-instance of RPP. Fix a placement of the pieces of $I'$ that forms $O^*$, and $\ell$ be one of its lines of symmetry. Assume without loss of generality that $\ell$ is a vertical line. Let $F^\ell$ be the mirror-symmetry of $F$ with respect to $\ell$. It can be shown that $\mathsf{area}(F \cap F^\ell) \geq WH - 2wh \geq 10w^2$, implying that $\ell$ passes through an interior point of $F$. Let $\ell_B$ be the line containing the segment of $F$ with length $4w$. Let $c$ be the center of the frame piece's bounding box.

**Lemma 2** *$\ell_B$ is either parallel or orthogonal to $\ell$. Further, $\ell$ passes through $c$.*

So $\ell$ passes through $c$ and is either parallel or orthogonal to $\ell_B$ (see Figure 4). In either case, $F \cup F^L$

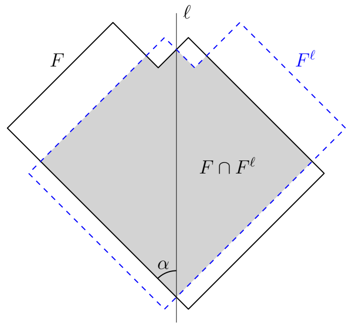


Figure 2: If $\ell$ and $\ell_B$ form an angle of $\pi/4$, then $F \cap F^\ell$ is contained in a rectangle in an $H \times H$ and thus $O^*$ cannot be mirror-symmetric.

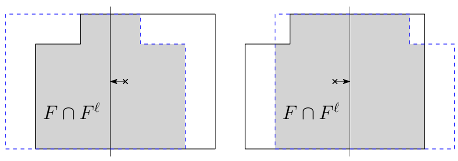


Figure 3: When $\ell$ passes to the left of $c$, the portion of $F$ to the left of $\ell$ is too small [Left]. If it passes to the right [Right], the right portion would be too small.

is a $W \times H$ rectangle, and thus $O^* = F \cup F^L$. This implies that $O^* \setminus F$ is a $w \times h$ rectangle which must contain the remaining pieces of $I'$. In particular, we have that this placement gives a solution to the instance $I$ of RPP, completing the proof of Theorem 1.

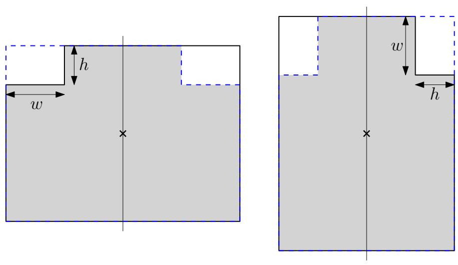


Figure 4: If $\ell$ passes through $c$, and is either orthogonal or parallel to $\ell_B$, the symmetric assembly puzzle can only be completed into a rectangle.

Next we turn to symmetric assembly puzzles with a constant number of polygonal pieces with many vertices, and show that this generalization can be solved in polynomial time.

**Theorem 3** *Symmetric assembly puzzles with a constant number of polygonal pieces, having a total of $n$ vertices, are polynomial-time (in $n$) solvable.*

The proof is a careful case analysis, focusing on the special case of two pieces, $P_1$ and $P_2$, which are simple polygons having $n_1$ and $n_2$ vertices (respectively). Detecting whether or not a simple $n$-gon has a line of symmetry can be done in time $O(n)$ [5]. For symmetric assembly puzzles, our goal is to determine if there exists a mirror-symmetric simple polygon $R$ such that $R$ can be exactly covered (without overlap) by $P_1$ and $P_2$, allowing them to be flipped and rotated.

Here, we consider $R$ to be a nondegenerate simple polygon, disallowing, e.g., that it have a cut vertex. The degenerate case can be handled easily, assuming that $P_1$ and $P_2$ are nondegenerate simple polygons (without cut vertices themselves), since the presence of a cut vertex at $v$ of $R$ implies that $v$ is a vertex of either $P_1$ or $P_2$ that is in contact with a vertex or edge of the other piece, and that any valid line of symmetry, $\ell$, must pass through $v$ (by the assumption that each piece $P_i$ has no cut vertex); thus, each of $P_1$ and $P_2$ would have to be mirror-symmetric, with at least one of them mirror-symmetric with respect to a line through one of its vertices. We then considers how the line of symmetry, $\ell$, lies with respect to $R$, $P_1$, and $P_2$; refer to Fig. 5 for a depiction of the cases; analysis not presented here.

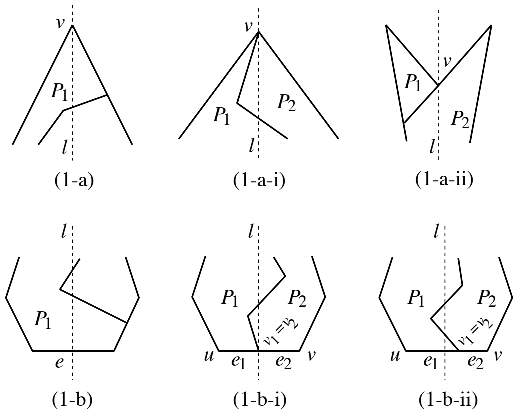


Figure 5: Case analysis for how the line of symmetry $\ell$ lies with respect to $R$, $P_1$, and $P_2$.

The main remaining open question is whether symmetric assembly puzzles are fixed-parameter tractable with respect to the number of pieces. We conjecture, however, that the problem is W[1]-hard.

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