Ununfoldable Polyhedra with Convex Faces

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Abstract

Unfolding a convex polyhedron into a simple planar polygon is a well-studied problem. In this paper, we study the limits of unfoldability by studying nonconvex polyhedra with the same combinatorial structure as convex polyhedra. In particular, we give two examples of polyhedra, one with 24 convex faces and one with 36 triangular faces, that cannot be unfolded by cutting along edges. We further show that such a polyhedron can indeed be unfolded if cuts are allowed to cross faces. Finally, we prove that "open" polyhedra with triangular faces may not be unfoldable no matter how they are cut.

1 Introduction

A classic open question in geometry [5, 12, 20, 24] is whether every convex polyhedron can be cut along its edges and flattened into the plane without any overlap. Such a collection of cuts is called an *edge cutting* of the polyhedron, and the resulting simple polygon is called an *edge unfolding* or *net*. While the first explicit description of this problem is by Shephard in 1975 [24], it has been implicit since at least the time of Albrecht Dürer, circa 1500 [11].

It is widely conjectured that every convex polyhedron has an edge unfolding. Some recent support for this conjecture is that every triangulated convex polyhedron has a *vertex unfolding*, in which the cuts are along edges but the unfolding only needs to be connected at vertices [10]. On the other hand, experimental results suggest that a random edge cutting of

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a random polytope causes overlap with probability approaching 1 as the number of vertices approaches infinity [22].

While unfoldings were originally used to make paper models of polyhedra [7, 27], unfoldings have important industrial applications. For example, sheet metal bending is an efficient process for manufacturing [14, 26]. In this process, the desired object is approximated by a polyhedron, which is unfolded into a collection of polygons. Then these polygons are cut out of a sheet of material, and each piece is folded into a portion of the object's surface using a bending machine. The unfoldings have multiple pieces partly for practical reasons such as efficient packing into a rectangle of material, but mainly because little theory on unfolding nonconvex polyhedra is available, and thus heuristics must be used [20].

There are two freely available heuristic programs for constructing edge unfoldings of polyhedra: the Mathematica package UnfoldPolytope [19], and the Macintosh program HyperGami [15]. There are no reports of these programs failing to find an edge unfolding for a convex polyhedron; HyperGami even finds unfoldings for nonconvex polyhedra. There are also several commercial heuristic programs; an example is Touch-3D [17], which supports nonconvex polyhedra by using multiple pieces when needed.

It is known that if we allow cuts across the faces as well as along the edges, then every *convex* polyhedron has an unfolding. Two such unfoldings are known. The simplest to describe is the *star unfolding* [1, 2], which cuts from a generic point on the polyhedron along shortest paths to each of the vertices. The second is the *source unfolding* [18, 23], which cuts along points with more than one shortest path to a generic source point.

There has been little theoretical work on unfolding nonconvex polyhedra. In what may be the only paper on this subject, Biedl et al. [4] show the positive result that certain classes of orthogonal polyhedra can be unfolded. They show the negative result that not all nonconvex polyhedra have edge unfoldings. Two of their examples are given in Figure 1. The first example is rather trivial: the top box must unfold to fit inside the hole of the top face of the bottom box, but there is insufficient area to do so. The second example is closer to a convex polyhedron in the sense that every face is homeomorphic to a disk.

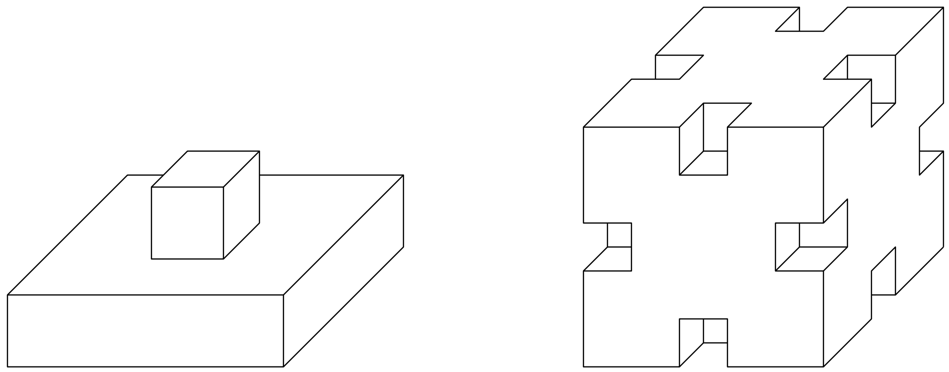

Figure 1: Orthogonal polyhedra with no edge unfoldings from [4].

Neither of these examples is satisfying because they are not "topologically convex." A polyhedron is *topologically convex* if its graph (1-skeleton) is isomorphic to the graph of a convex polyhedron. The first example of Figure 1 is ruled out because faces of a convex polyhedron are always homeomorphic to disks. The second example is ruled out because there are pairs of faces that share more than one edge, which is impossible for a convex polyhedron[1]. In general, a famous theorem of Steinitz [13, 16, 25] tells us that a polyhedron is topologically convex precisely if its graph is 3-connected and planar.

The class of topologically convex polyhedra includes all *convex-faced* polyhedra (i.e., polyhedra whose faces are all convex) that are homeomorphic to spheres. Schevon and other researchers [4, 21] have asked whether all such polyhedra can be unfolded without overlap by cutting along edges. In other words, can the conjecture that every convex polyhedron is edge-unfoldable be extended to topologically convex polyhedra? Another particularly interesting subclass, which we consider here, are polyhedra whose faces are all triangles (called *triangulated* or *simplicial*) and are homeomorphic to spheres.

In this paper we construct families of triangulated and convex-faced polyhedra that are homeomorphic to spheres and have no edge unfoldings. This proves, in particular, that the edge-unfolding conjecture does not generalize to topologically convex polyhedra. We go on to show that cuts across faces can unfold some convex-faced polyhedra that cannot be unfolded with cuts only along edges. This is the first demonstration that general cuts are more powerful than edge cuts.

We also consider the problem of constructing a polyhedron that cannot be unfolded even using general cuts. If such a polyhedron exists, the theorem that every convex polyhedron is generally unfoldable (using, for example, the star or source unfolding) cannot be extended to topologically convex polyhedra. As a step towards this goal, we present an "open" triangulated polyhedron that cannot be unfolded. Finding a "closed" ununfoldable polyhedron is an intriguing open problem.

A preliminary version of this work appeared in CCCG'99 [3].

2 Basics

We begin with formal definitions and some basic results about polyhedra, unfoldings, and cuttings.

We define a *polyhedron* to be a connected set of closed planar polygons in 3-space such that (1) any intersection between two polygons in the set is a collection of vertices and edges common to both polygons, and (2) each edge is shared by at most two polygons in the set. A *closed* polyhedron is one in which each edge is shared by exactly two polygons. If a polyhedron is not closed, we call it an *open* polyhedron; the *boundary* of an open polyhedron is the set of edges covered by only one polygon.

In general, we may allow the polygonal faces to be multiply connected (i.e., have holes). However, in this paper we concentrate on convex-faced polyhedra. A polyhedron is *convex-faced* if every face is strictly convex, that is, every interior face angle is strictly less than π. Thus, in particular, every face of a convex-faced polyhedron is simply connected. A polyhedron is *triangulated* if every face is a triangle, that is, has three vertices.

Next we prove that convex-faced polyhedra (and hence triangulated polyhedra) are a

[1]We disallow polyhedra with "floating vertices," that is, vertices with angle π on both incident faces, which can just be removed.

subclass of *topologically convex* polyhedra, that is, polyhedra whose graphs (1-skeleta) are isomorphic to graphs of convex polyhedra. A *convex polyhedron* is a closed polyhedron whose interior is a convex set—equivalently, the open line segment connecting any pair of points on the polyhedron's surface is interior to the polyhedron.

Theorem 1 (Steinitz's Theorem [13, 16, 25]**)** *A graph is the graph of a convex polyhedron precisely if it is 3-connected and planar.*

Corollary 2 *Every convex-faced closed polyhedron that is homeomorphic to a sphere is topologically convex.*

Proof: Because the polyhedron is homeomorphic to a sphere, its graph G must be planar. It remains to show that G is 3-connected. If there is a vertex v whose removal disconnects G into at least two components G_1 and G_2, then there must be a "belt" wrapping around the polyhedron that separates G_1 and G_2. This belt has only one vertex (v) connecting G_1 and G_2, so it must consist of a single face. This face touches itself at v, which is impossible for a convex polygon. Similarly, if there is a pair (v_1, v_2) of vertices whose removal disconnects G into at least two components G_1 and G_2, then again there must be a "belt" separating the two components, but this time the belt connects G_1 and G_2 at two vertices (v_1 and v_2). Thus, the belt can consist of up to two faces, and these faces share two vertices v_1 and v_2. This is impossible for strictly convex polygons. □

A *cutting* of a polyhedron P is a union C of a finite number of line segments (called *cuts*) on P, such that cutting along C results in a connected surface $P - C$ that can be flattened into the plane (that is, isometrically embedded) without overlap. The resulting flattened form is called an *unfolding* of P. An *edge cutting* of P is a cutting of P that is just a union of P's edges. The corresponding unfolding is called an *edge unfolding*. We sometimes call unfoldings *general unfoldings* to distinguish them from edge unfoldings. We call a polyhedron *unfoldable* if it has a general unfolding, and *edge-unfoldable* if it has an edge unfolding. Similarly, we call a polyhedron [*edge-*]*ununfoldable* if it is not [edge-]unfoldable.

If P is an open polyhedron, we limit attention to cuttings that contain only a finite number of boundary points of P. Every neighborhood of a boundary point of P in C must contain an interior point of P in C; otherwise, the boundary point could be removed from C without changing $P - C$. Hence, the collection of boundary points of P in C can be reduced down to a finite set without any effect.

We define the *curvature* of an interior vertex v to be the discrete analog of Gaussian curvature, namely 2π minus the sum of the face angles at v. (For vertices on the boundary of the polyhedron, we do not define the notion of curvature.) Hence, the neighborhood of a zero-curvature vertex can be flattened into the plane, the neighborhood of a positive-curvature vertex (for example, a cone) requires a cut in order to be flattened, and the neighborhood of a negative-curvature vertex (for example, a saddle) requires two or more cuts to avoid self-overlap.

We look at a cutting as a graph drawn on the polyhedron, and establish some basic facts about these graphs in the next lemmas.

Lemma 3 *Any cutting of a polyhedron P is a forest, and spans every nonboundary vertex of P that has nonzero curvature.*

Proof: If a cutting contained a cycle, having positive area both interior and exterior to the cycle, then the resulting unfolding would be disconnected, a contradiction. (Above we excluded the possibility of the cutting containing a cycle on the boundary of an open polyhedron P.) If the cutting did not contain a particular (nonboundary) point with nonzero curvature, neighborhoods of that point could not be flattened without overlap. $\square$

A common assumption is that an unfolding must be a simple polygon, which implies that a cutting of a closed polyhedron must furthermore be a (connected) tree; see e.g. [8, 9, 22]. Without this restriction, in most cases, a cutting of a closed polyhedron is a tree, but in fact this is not always the case. Figure 2 illustrates a basic construction for separating off a connected component of the cutting. We could add cuts to connect the inner tree to the rest of the forest, but these extra cuts are unnecessary. Using this construction, we can build a polyhedron and a cutting having arbitrarily many connected components, by connecting a series of the constructions in Figure 2 in a "dented barrel" shape, and then capping the ends.

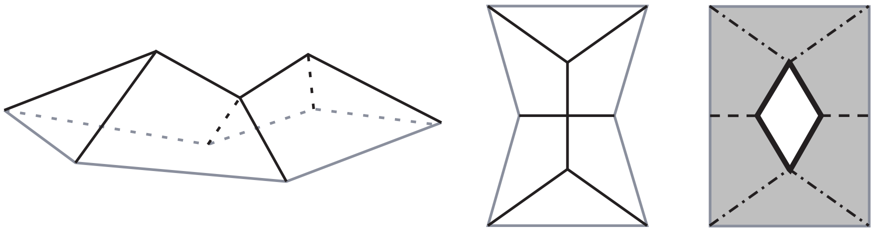

Figure 2: A portion of a polyhedron in (left) perspective view and (middle) birds-eye view, and (right) the unfolding that results from a cutting with a separated connected component of cuts. To formally construct this portion of a polyhedron, start with the unfolding on the right, and fold as shown.

Together with Joseph O'Rourke (personal communication, July 2001), we have proved that such examples require nonconvexity:

Lemma 4 *Any cutting of a convex closed polyhedron is a tree.*

Proof: By Lemma 3, the cutting is a forest. Suppose for contradiction that the forest has multiple connected components. Then there is a closed path p on the surface of the polyhedron that avoids all cuts and strictly encloses a connected component of the cutting. In particular, p avoids all vertices of the polyhedron. Let τ denote the total turn angle along the path p. Because p avoids vertices, it unfolds to a connected (uncut) closed path in the planar layout; thus, $\tau = 2\pi$. Because the polyhedron is homeomorphic to a sphere, the Gauss-Bonnet theorem [6, pp. 215–217] applied to p says that $\tau + \gamma = 2\pi$ where γ is the curvature enclosed by p. By convexity, $\gamma \geq 0$. Further, p encloses a connected component of the cutting, which consists of at least one vertex of the polyhedron, so $\gamma > 0$. Therefore, $\tau < 2\pi$, contradicting that p could lay out flat in the plane. $\square$

Lemma 5 *If v is a vertex of a polyhedron P with negative curvature, then any cutting of P must include more than one cut incident to v.*

Proof: Suppose some cutting C includes only a single cut incident to v. Let N be $P \cap B$, where B is a small ball around v. Neighborhood N unfolds to a small disk that self-overlaps by precisely the absolute value of the curvature of v. □

3 Hats

Our construction of a closed polyhedron with no edge unfolding begins by constructing open polyhedra that cannot be edge unfolded. The intuition is to build a "hat-shaped" polyhedron having just one interior vertex with positive curvature (the peak of the hat). The remaining vertices have negative curvature and this severely limits the possible edge cuttings.

We know of two combinatorially different families of convex-faced *hats* that suffice to prevent edge cuttings. They are shown in Figures 3 and 4. The remainder of this section defines and analyzes hats in detail. These hats have the additional property that their boundary is a triangle, and in the next section we will exploit this by gluing each hat to a face of a regular tetrahedron.

The first hat (Figure 3) is called a *basic hat* because of its low face count. The second hat (Figure 4), however, has only triangular faces, and is hence called a *triangulated hat*. In both cases, we will eventually find a convex-faced closed polyhedra with no edge unfoldings, and in the latter case it will also be triangulated.

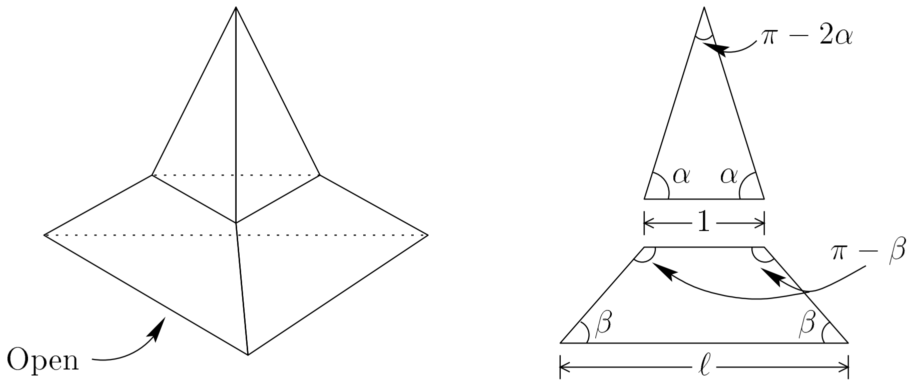

Figure 3: A basic hat and its constituent faces.

The two types of hats are similar, so we will treat them together. First we need some terminology for features of the hats. The three boundary vertices are called the *corners* of the hat, the innermost vertex is the *tip*, and the remaining vertices are the *middle* vertices. The three triangles incident to the tip form the *spike* of the hat. The remaining faces (incident to the corners) form the *brim* of the hat. The brim of a basic hat consists of three trapezoids, whereas the brim of a triangulated hat consists of six triangles. Note that the spike of the triangulated hat is rotated 60° relative to the boundary, in contrast to the basic hat.

More formally, hats are parameterized by three parameters. Basic hats are parameterized by angles α and β satisfying $30° \leq \alpha, \beta < 90°$, and a length $\ell > 1$. For consistency with the other type of hat, we also define $\gamma = 0$ for basic hats. The spike is an open tetrahedron consisting of three identical isosceles triangles, where the lower angles are α and the bottom

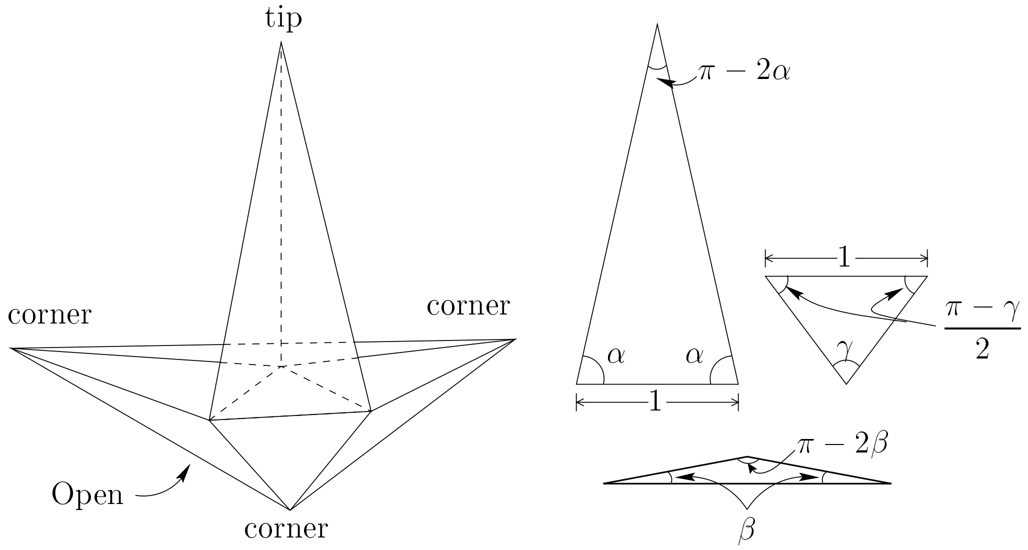

Figure 4: A triangulated hat and its constituent faces.

sides have unit length. The brim is an open truncated tetrahedron consisting of three identical symmetric trapezoids, where the lower angles are β, the top length is 1, and the bottom length is ℓ.

Triangulated hats are parameterized by angles α, β, and γ satisfying $30^\circ \leq \alpha, \beta + \gamma/2 < 90^\circ$ and $\gamma < 60^\circ$. The spike is an open tetrahedron with base angles α, just like the basic hat. In the brim, there are three identical isosceles triangles touching the boundary along their base edges, which have base angles β; and three identical isosceles triangles touching the peak at their base edges, and touching the boundary at their opposite vertex, whose angle is γ.

As mentioned above, the key to our constructions is to make the middle vertices have negative curvature.

Lemma 6 *The middle vertices of a hat have negative curvature precisely if $\alpha > \beta + \gamma/2$. In particular, for any valid choice of β and γ, there is a valid choice of α that satisfies this property.*

Proof: The first claim follows simply by summing the angles incident to a middle vertex and checking when that sum is greater than 2π. For basic hats, we have

$$2\alpha + 2(\pi - \beta) > 2\pi$$

and for triangulated hats, we have

$$2\alpha + 2\frac{\pi - \gamma}{2} + \pi - 2\beta > 2\pi.$$

Now α is only restricted by $30^\circ \leq \alpha < 90^\circ$ for validity and $\alpha > \beta + \gamma/2$ for negative curvature. Because $30^\circ \leq \beta + \gamma/2$, these two conditions are satisfied precisely if $\beta + \gamma/2 < \alpha < 90^\circ$, which is satisfiable because $\beta + \gamma/2 < 90^\circ$. $\square$

Note that these conditions are rather symmetric for the two types of hats, specifying that the angle at a base vertex of the spike is larger than the angle at a corner of the hat.

A more useful property of hats is that the middle vertices can be made to have negative curvature even when one of the spike triangles is cut away:

Lemma 7 *The middle vertices of a hat have negative curvature even when a spike triangle is removed, provided* $\alpha > 2\beta + \gamma$*. In particular, for any* β *and* γ *with* $30° \leq \beta + \gamma/2 < 45°$ *and* $\gamma < 60°$*, there is a valid choice of* α *that satisfies these properties.*

Proof: The first claim follows from Lemma 6 by halving the coefficient of α, because only a single α is now included in the sum of angles. The constraints now become $30° \leq \alpha < 90°$ and $\alpha > 2\beta + \gamma$. Because $30° \leq \beta + \gamma/2$, these are equivalent to $2\beta + \gamma < \alpha < 90°$, which is achievable because $\beta + \gamma/2 < 45°$. □

For example, a hat with angles $\alpha = 81°$, $\beta = 30°$, and $\gamma = 20°$, has negative curvature at the middle vertices even when a spike triangle is removed. In the case of the basic hat, we would have $\alpha + 2(\pi - \beta) = 381°$, and for the triangulated hat $\alpha + (\pi - 2\beta) + 2(\pi - \gamma)/2 = 361°$.

Theorem 8 *Hats that satisfy the constraints in Lemma 7 are open convex-faced polyhedra with no edge unfoldings.*

Proof: By Lemma 3, any edge cutting is a forest of nonboundary edges that covers the tip and middle vertices. Every connected component of the cutting is a tree, and so must have at least two leaves. Note that no two corners of the hat can be leaves of a common connected component of the cutting, because otherwise the path of cuts connecting them would disconnect the polyhedron. (Recall we excluded the possibility of a boundary edge being in a cutting.) Thus, at most one corner is a leaf of each connected component of the cutting. By Lemma 6, the middle vertices have negative curvature, so by Lemma 5, they cannot be leaves of the cutting. Hence, the cutting must in fact be a single path from a corner to the tip, visiting all of the middle vertices.

It is possible to argue by case analysis that, for basic hats, there is precisely one such path up to symmetry (see Figure 5, left), and for triangulated hats, there are two such paths up to symmetry (see Figure 5, center and right). Each of these cuttings has a vertex with only one spike triangle cut away (marked by a gray circle in Figure 5), which means the remainder has negative curvature, leading to overlap by Lemma 5.

However, this can be argued more simply as follows. Because the spike remains connected to the rest of the polyhedron, there must be a spike triangle A that remains connected to a brim face B; see Figure 6. Because there is only one cut in the hat incident to a corner, this cut is not incident to one of the two vertices shared by faces A and B, say v. Therefore the brim faces B, C, and (in the case of a triangulated hat) D incident to v and the spike face A remain connected in the unfolding along edges incident to v. But by Lemma 7, these faces have total angle at v of more than $360°$, causing overlap. □

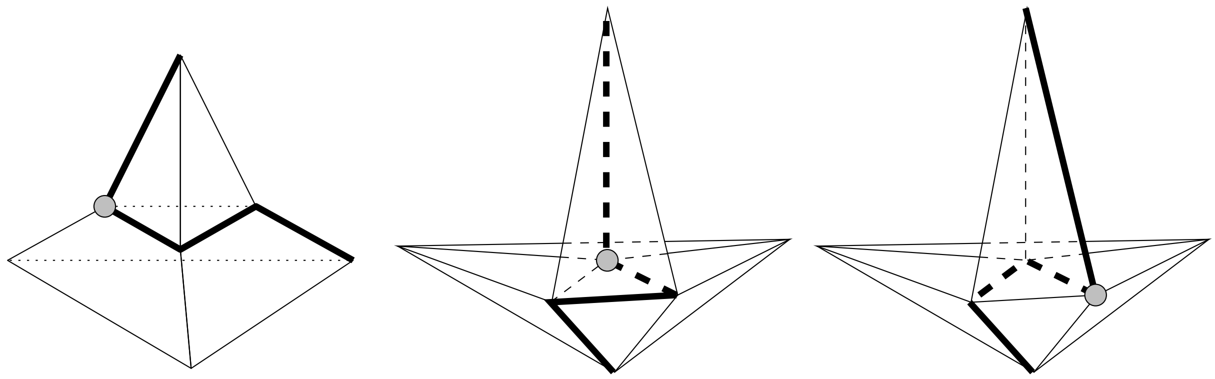

Figure 5: The possible cuttings (up to symmetry) of a hat satisfying Lemma 7.

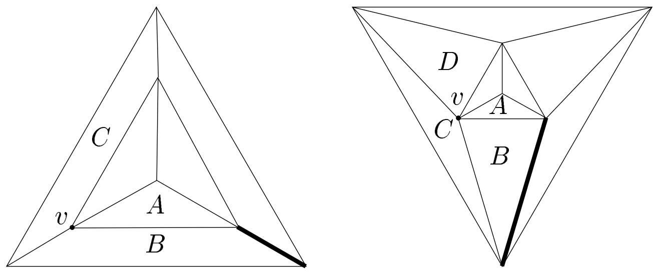

Figure 6: Proof of Lemma 8: a planar drawing of each kind of hat with a possible cut in bold.

4 Gluing Hats Together

Because the boundary of a hat is an equilateral triangle, we can take four hats and place them against the faces of a regular tetrahedron (and then remove the guiding tetrahedron). The result is a closed polyhedron with no edge unfolding, which we call a *spiked tetrahedron*. First observe the following property of unfolded hats:

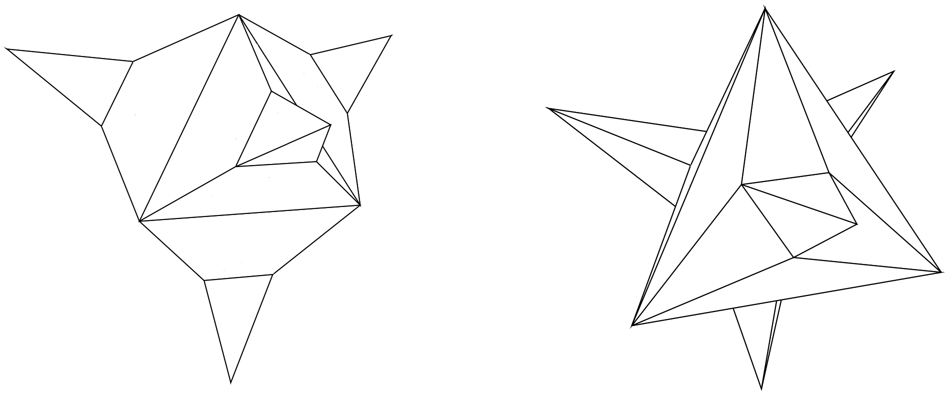

Figure 7: Spiked tetrahedra for both types of hats.

Lemma 9 *In any edge cutting of a spiked tetrahedron, there is a path joining at least two corners of each hat using nonboundary edges of that hat, provided the parameters satisfy the constraint in Lemma 7.*

Proof: This lemma follows directly from Theorem 8: an edge cutting of the spiked tetrahedron cannot induce an edge cutting of a constituent hat, and cannot cause overlap, so it must cut each hat into at least two pieces, by way of a path with the claimed properties. □

Now the desired result follows easily:

Theorem 10 *Spiked tetrahedra are convex-faced closed polyhedra with no edge unfoldings, provided the constituent hats satisfy the constraint in Lemma 7.*

Proof: Suppose there were an edge cutting. By Lemma 9, inside each of the four hats would be paths of cuts joining two corners, and these paths share no cuts because they use only nonboundary edges. But because there are only four corners in total, these paths would form a cycle in the cutting, contradicting Lemma 3. □

This theorem also proves that the analogously defined *spiked octahedron* cannot be edge unfolded for hats satisfying the constrain in Lemma 7. In particular, there is a basic-hat spiked octahedron having no edge unfolding, for any β satisfying $30° \leq \beta < 45°$. Using a different proof technique [3], it can be shown that there is such a spiked octahedron for any β satisfying $45° < \beta < 60°$.

5 General Unfolding

While spiked tetrahedra are not edge-unfoldable for certain choices of the parameters, they are always generally unfoldable; see Figure 8. We start with the parallelogram unfolding of the underlying tetrahedron on which we glued the hats. The spikes do not have room to unfold in the middle of an unfolded brim, so we use the following trick. Cut out each spike and a small band that connects it to an edge on the boundary of the tetrahedron unfolding. Now attach that band and unfolded spike to the corresponding edge of the tetrahedron unfolding (corresponding in the sense that the edges are glued to each other). The bands are chosen to be nonperpendicular so that a band does not attempt to attach where another band was removed.

6 No General Unfolding

An intriguing open question is whether there is a convex-faced polyhedron, triangulated polyhedron, or any closed polyhedron that cannot be generally unfolded. This section makes a step toward solving this problem by presenting a triangulated *open* polyhedron (polyhedron with boundary) with no general unfolding.

The construction is to connect several triangles in a cycle, all sharing a common vertex v, as shown in Figure 9. By connecting enough triangles and/or adjusting the triangles to have large enough angle incident to v, we can arrange for vertex v to have negative curvature.

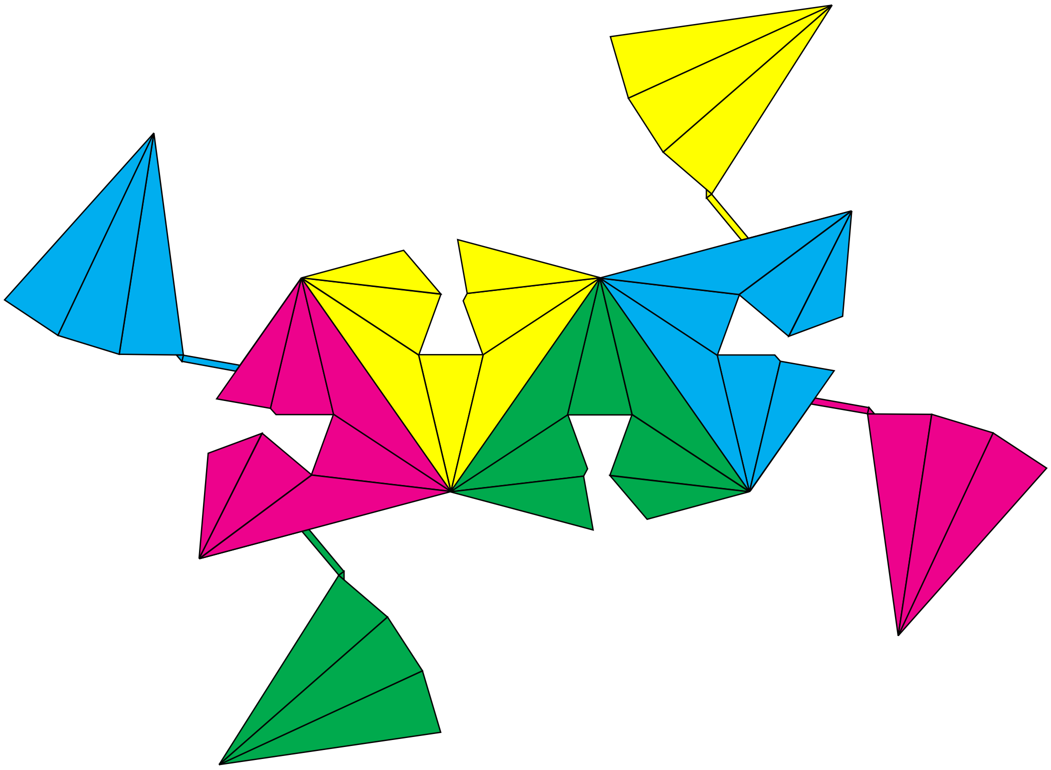

Figure 8: General unfolding of a spiked tetrahedron.

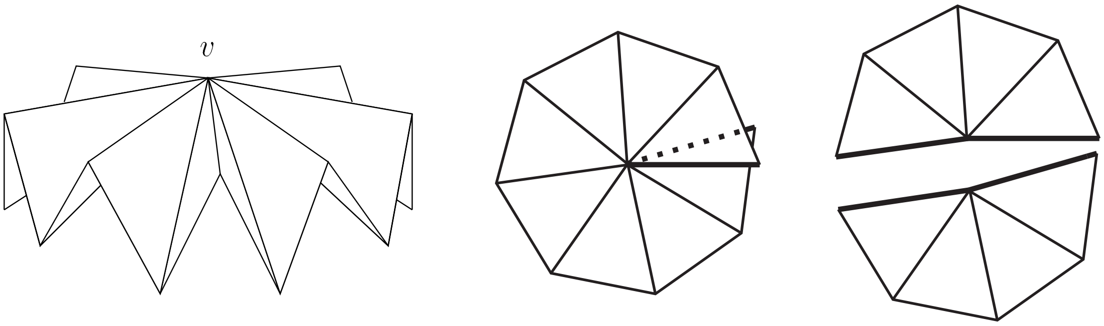

Figure 9: Open polyhedron with no unfolding. One cut creates an unfolding with overlap, but two cuts disconnect the results.

Theorem 11 *The open polyhedron in Figure 9 has no general unfolding if v has negative curvature.*

Proof: A cutting could only have leaves on the boundary, because v has negative curvature, and because the cut incident to any other leaf could be glued (uncut) without affecting the unfolding. But any cutting has at least two leaves, so it must disconnect the polyhedron, a contradiction. □

7 Conclusion

The spiked tetrahedra (Figure 7) show that the conjecture about edge unfoldings of convex polyhedra cannot be extended to topologically convex polyhedra. Figure 8 further illustrates the added power of cuts along faces for topologically convex polyhedra. Figure 9 shows an open polyhedron that cannot be unfolded at all, but the flexibility exploited in Figure 8 suggests that it will be more difficult to settle whether there is a closed polyhedron with that property. Another interesting open question is the complexity of deciding whether a given triangulated polyhedron has an edge unfolding, now that we know that the answer is not always "yes."

Acknowledgments

We thank Anna Lubiw for helpful discussions. We thank Joseph O'Rourke for his contributions to Section 2 and for many helpful comments on the paper.

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