Zipper Unfolding of Domes and Prismoids

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Abstract

We study Hamiltonian unfolding—cutting a convex polyhedron along a Hamiltonian path of edges to unfold it without overlap—of two classes of polyhedra. Such unfoldings could be implemented by a single zipper, so they are also known as zipper edge unfoldings. First we consider domes, which are simple convex polyhedra. We find a family of domes whose graphs are Hamiltonian, yet any Hamiltonian unfolding causes overlap, making the domes Hamiltonian-ununfoldable. Second we turn to prismoids, which are another family of simple convex polyhedra. We show that any nested prismoid is Hamiltonian-unfoldable, and that for general prismoids, Hamiltonian unfoldability can be tested in polynomial time.

Keywords: edge unfolding, Hamiltonian-unfolding, zipper unfolding, paper folding, dome, prismoid.

1 Introduction

A common way to make a polyhedron from paper is to fold and glue a planar polygonal shape, called a *net* of the polyhedron. The characterization of polyhedra and their nets has been investigated since Dürer used nets to represent polyhedra in his 1525 book (see [DO07, O'R11]). One long-standing open problem is whether every convex polyhedron can be developed into a flat nonoverlapping polygonal shape by cutting only along its edges. Such a development is called an *edge unfolding* of the polyhedron. So far, very special classes of edge-unfoldable convex polyhedra are known: polyhedra of at most six vertices [DiB90], pyramids, prisms, prismoids, and domes [O'R01, DO07, O'R08].

In any edge unfolding, the cut edges produce a spanning tree of the graph representing the combinatorial structure of the convex polyhedron. One possible approach to the open problem is to restrict the cutting spanning tree to be a simple path. Because the path should visit (or cut) every vertex exactly once, the cutting edges produce a Hamiltonian path along the edges of the polyhedron. This restricted type of edge unfolding is called a *Hamiltonian unfolding* [DDL$^+$10]. From an industrial point of view, such an unfolding can be realized by a zipper, and there are several products based on this idea (Figure 1).

Figure 1: Zipper folding bags (Top: Spiral Wristlets (http://www.cathayana.com/su13.htm). Bottom: ZipIt Monster (http://www.zipitstore.com))

From the graph-theoretical point of view, the Hamiltonian unfolding problem is related to the Hamiltonian path problem on a graph representing the vertices and the edges of the polyhedron. More precisely, if a polyhedron is Hamiltonian-unfoldable, then its corresponding graph must have a Hamiltonian path. Recently, Demaine et al. [DDL$^+$10] found that all Archimedean solids are Hamiltonian-unfoldable. On the other hand, a rhombic dodecahedron does not have a Hamiltonian-unfolding because its corresponding graph has no Hamiltonian path [DDL$^+$10]. As far as the authors know, all Hamiltonian-*un*unfoldable polyhedra have been proved in this combinatorial way, by showing that their corresponding graphs are not Hamiltonian.

On the other hand, the difficulty of edge unfolding convex polyhedron comes from the fact that we have

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no general strategy to check whether its development causes an overlap no matter how it is cut along its edges. That is, to solve the open problem negatively, we have to find a convex polyhedron that causes an overlap by edge unfolding along any spanning tree. In this sense, a natural question arises: is there a convex polyhedron whose corresponding graph has a Hamiltonian path, yet any Hamiltonian unfolding causes overlap?

Our results. Our first result is an affirmative answer to the natural question. We show a family of convex polyhedra, which are simple domes, such that an overlap occurs in every Hamiltonian unfolding. Each of our domes has many Hamiltonian paths on its corresponding graph. Thus we can say that a graph-theoretic approach is not enough to tackle the open problem even for quite simple convex polyhedra.

Extending this result, for any fixed integer k, we show that there exists a family of domes that cannot be edge-unfolded by any cutting tree of degree at most k. That is, we show that, if the degree of the spanning tree of cuts is bounded, there exist infinitely many convex polyhedra that cannot be edge-unfolded. Hamiltonian unfoldings are the special case when the degree bound k is 2.

Next we turn to prismoids; a *prismoid* is the convex hull of two parallel convex polygons whose corresponding angles are equal. If one of these polygons contains the other in the projection orthogonal to the parallel planes, then the prismoid is *nested*. We give positive results about prismoids. First we show that any nested prismoid can be unfolded by a Hamiltonian unfolding. This result is based on band unfolding of nested prismoids developed in [ADL+08]. Second we show how to determine whether a general prismoid can be Hamiltonian-unfolded in polynomial time. This result is based on counting of the number of Hamiltonian paths of a general prismoid. We conjecture that any (general) prismoid can be Hamiltonian-unfolded, but this problem remains unsolved.

2 Hamiltonian-Unfoldable Dome

For any integer $n \geq 3$, a *dome* is a convex polyhedron that consists of a (convex) polygonal *base*, and n (convex) polygonal *sides*, each of which shares a distinct edge with the base (see, e.g., [DO07]).

First we state a technical lemma.

Lemma 1 *For a positive integer n, let $\theta = \frac{2\pi}{n}$. Let T be an isosceles triangle with apex angle θ. Two arms of T are of unit length. We place eight copies of T as in Figure 2, where bold edges are shared by two triangles. Then the triangles T_4 and T_8 overlap for any $n > 12$.*

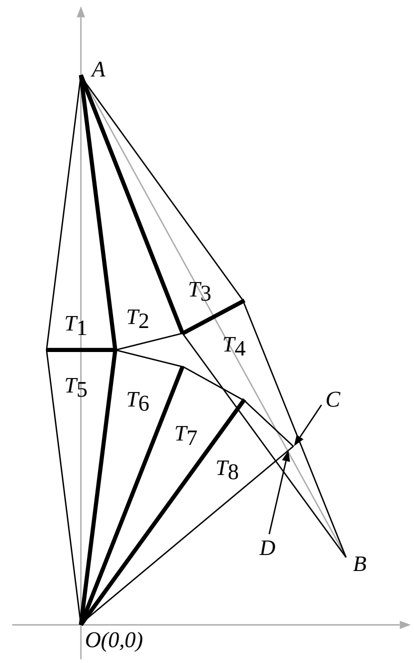

Figure 2: Overlapping triangles

Proof. We put the origin $O = (0,0)$ on the apex of T_5, and the y axis on the line joining the apices of T_1 and T_5. Let A and B be the apices of T_1 and T_4, respectively. Then we can compute

$$A = \left(0, 2\sin\tfrac{\theta}{2}\right), B = \left(2\sin\tfrac{\theta}{2}\sin 2\theta, 2\sin\tfrac{\theta}{2}(1-\cos 2\theta)\right).$$

On the other hand, let C be the furthest base angle point of T_8 from T_5. Then we have

$$C = \left(\cos\tfrac{7\theta}{2}, \sin\tfrac{7\theta}{2}\right).$$

Now consider the intersection point D on two lines AB and OC. (Precisely, two lines containing AB and OC.) Then both T_4 and T_8 contain the point D if $|OD| < 1$. By a simple computation, we obtain

$$D = \left(\frac{2\sin\frac{\theta}{2}}{\cot 2\theta + \cot\frac{7\theta}{2}}, \frac{2\sin\frac{\theta}{2}\tan 2\theta}{\tan 2\theta + \tan\frac{7\theta}{2}}\right)$$

and hence $|OD|^2$ equals

$$4\sin^2\frac{\theta}{2}\left(\frac{1}{\left((\cot 2\theta + \cot\frac{7\theta}{2}\right)^2} + \frac{\tan^2 2\theta}{(\tan 2\theta + \tan\frac{7\theta}{2})^2}\right),$$

which is less than 1 for any $n > 12$. □

Theorem 2 *There exists an infinite sequence of domes that are Hamiltonian-ununfoldable.*

Proof. For each integer $n > 1$, we construct a dome $D(n)$ as follows. The base $B(n)$ is a regular $2n$-gon.

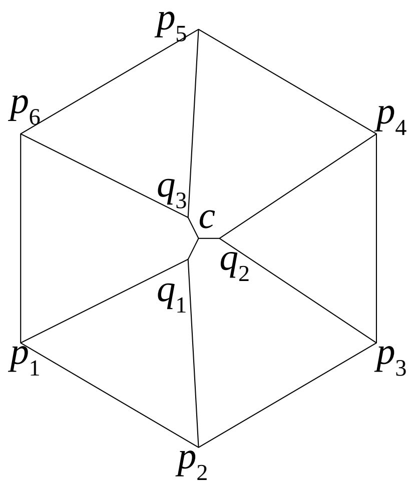

Figure 3: The top view of $D(3)$

Let $p_1, p_2, \ldots, p_{2n}$ be the vertices of $B(n)$. The dome $D(n)$ has an apex c that is on the central perpendicular of $B(n)$. The height of c is very small. We put a small circle C centered at c, and put n points $q_1, q_2, \ldots, q_n$ on C such that these points form a regular n-gon. To simplify, we assume that the height of c and the radius of C are almost 0. Then we join and make edges $\{p_{2i-1}, q_i\}$ and $\{p_{2i}, q_i\}$ for each $i = 1, 2, \ldots, n$. We rotate the circle C so that each triangle $q_i p_{2i-1} p_{2i}$ is an isosceles triangle. We also join c to the q_i for each $i = 1, 2, \ldots, n$. Figure 3 shows the top view of the dome $D(n)$ for $n = 3$. Now we show that $D(n)$ is Hamiltonian-ununfoldable for $n > 12$.

Suppose that $D(n)$ is Hamiltonian-unfoldable by cutting along edges in P. Then P is a Hamiltonian path on $D(n)$. For vertex v we use $\deg_P(v)$ to denote the number of edges incident to v in P. That is, $\deg_P(v) = 1$ for two endpoints and $\deg_P(v) = 2$ for the other vertices because P is a Hamiltonian path. Thus $\deg_P(c)$ is one or two, and almost all vertices q_i have $\deg_P(q_i) = 2$. This implies that for almost all vertices q_i, the path (p_{2i-1}, q_i, p_{2i}) is a part of P. That is, most isosceles triangles will be flipped along their base lines like petals of a flower.

We have two cases. First, we suppose that c is an endpoint of P. Without loss of generality, we can assume that the path (c, q_1, p_1) is in P. Then, because c has no other cut except along (c, q_1), P contains all subpaths (p_{2i-1}, q_i, p_{2i}) with $1 < i \leq n$ (except $i = 1$). Then we have only two possible ways to make a Hamiltonian path. One is $(c, q_1, p_1, p_{2n}, q_n, p_{2n-1}, \ldots, p_4, q_2, p_3, p_2)$, and the other one is $(c, q_1, p_1, p_2, p_3, q_2, p_4, \ldots, p_{2n-1}, q_n, p_{2n})$.

The first subcase is illustrated in Figure 4(b) and (c). We first cut along the dotted path in Figure 4(b). Then we flip the *lid*, which consists of all pentagons and one

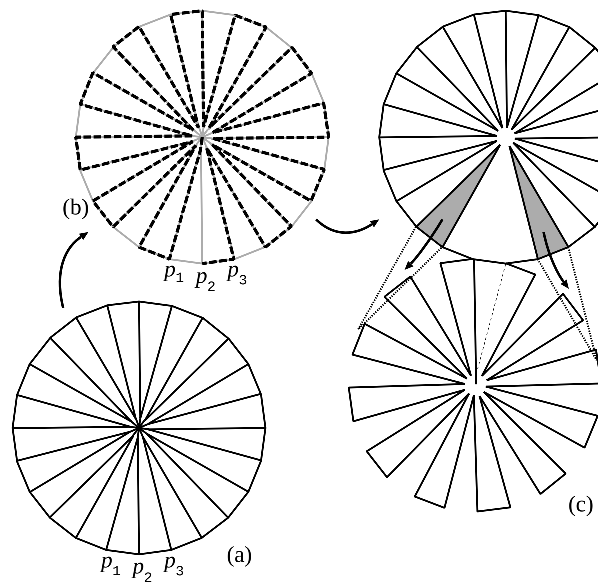

Figure 4: One possible development of $D(12)$

triangle $p_1p_2q_1$ (Figure 4(c)). Now the other triangles have to be flipped, however, the gray triangles overlap with the lid by Lemma 1 if the circle C and the height of the dome are sufficiently small and $n > 12$. Therefore, we cannot develop in this case without overlap. The second subcase is easier: one triangle closer to the lid again overlaps the flipped lid. Therefore, when c is an endpoint of P, every development causes an overlap.

Now we turn to the next case: c is not an endpoint of P. We now assume that the path (q_i, c, q_1, p_1) is in P without loss of generality for some i. When q_i is an endpoint, almost the same argument as the first case works. If $q_i = q_2$ or $q_i = q_n$, one of two petals overlaps, but in other cases, two petals again overlap the flipped lid. Therefore, we consider the case (p_j, q_i, c, q_1, p_1), where $j = 2i - 1$ or $j = 2i$. If we remove the vertices $\{p_j, q_i, c, q_1, p_1\}$ from the graph obtained from the dome $D(n)$, it is easy to see that the graph is disconnected into two parts. We call the graph induced by $\{p_2, p_3, \ldots, p_{j-1}, q_2, q_3, \ldots, q_{i-1}\}$ the *right* graph, and the other graph induced by $\{p_{j+1}, p_{j+1}, \ldots, p_{2n}, q_{i+1}, \ldots, q_n\}$ the *left* graph. Then, clearly, P consists of three parts; P_r for the right graph, P_l for the left graph, and the subpath (p_j, q_i, c, q_1, p_1) joining P_r and P_l. Now we take the larger graph P' between P_r and P_l, apply the same argument as the first case on P' with (p_j, q_i, c, q_1, p_1), and again obtain an overlap. $\square$

In the Hamiltonian unfolding, each vertex has degree at most 2 on the cutting path. This can be generalized to any integer $k \geq 2$:

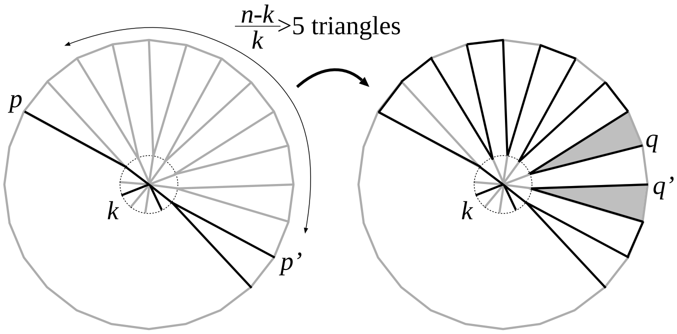

Figure 5: Maximum degree bounded case

Theorem 3 *For any positive integer $k \geq 2$, there exists an infinite number of domes that are edge-ununfoldable when the maximum degree of the cutting tree at each vertex is bounded at most k.*

We note that all vertices of the dome $D(n)$ have degree 3 except the central vertex c. That is, the cutting tree in Theorem 3 has only one vertex of degree greater than 3.

Proof. We consider the dome $D(n)$ for any $n > 6k$. Let T be any spanning tree of $D(n)$ with maximum degree at most k. We show that the development of $D(n)$ by cutting the edges in T causes an overlap. By definition, the central vertex c has degree at most k. Let T_c be the subtree of T induced by the vertices $\{c\} \cup N_T(c) \cup N_T(N_T(c))$, where $N_T(v)$ is the neighbor set of v on T, and $N_T(N_T(c)) = \cup_{q \in N_T(c)} N_T(q)$. Then, T_c has at most $2k$ leaves because each q_i may have two leaves from p_{2i-1} and p_{2i} in T_c. However, by the expected value argument, we have at least $(n-k)/k > 5$ consecutive triangles on the boundary of the base between two leaves p and p' of T_c (Figure 5). They are cut along T as was done in the proof of Theorem 2. Precisely, all pentagons between p and p' form a lid, and it is then flipped at one boundary edge, say $\{q, q'\}$ (Figure 5). When the triangles between p and p' are flipped, two triangles sharing q and q' (gray triangles in Figure 5) will overlap with the lid by Lemma 1. □

3 Hamiltonian-Unfoldability of a Prismoid

A *prismoid* is a convex hull of two parallel convex polygons with matching angles. If one of these polygons contains the other in the projection orthogonal to the parallel planes, the prismoid is *nested*. In a nested prismoid, the larger polygon is called the *base* and the other polygon is called the *top*. In general prismoids, we arbitrary name the two parallel convex polygons base and top. The other surface is called the *band*. Because the top and base have matching angles with parallel edges, the band consists of trapezoids.

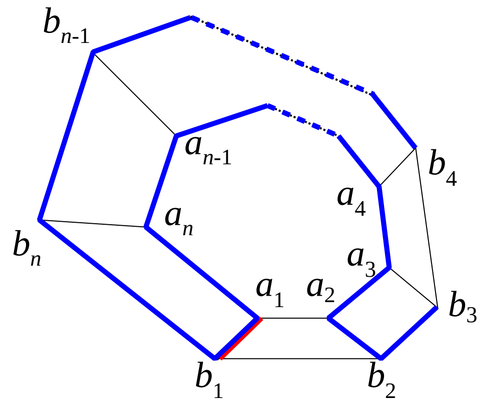

Figure 6: Hamiltonian-unfolding of a nested prismoid (1): (a_1, b_1) is the edge allowing us to unfold the band, and b_3b_4 is the first "acute" edge from b_1b_2.

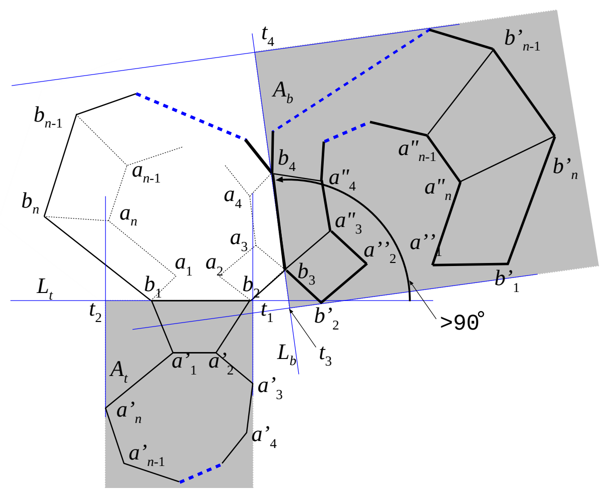

Figure 7: Hamiltonian-unfolding of a nested prismoid (2): the top and the band is fliped and separated from the base by the lines L_t and L_b, respectively.

3.1 Nested Prismoid

Theorem 4 *Any nested prismoid has a Hamiltonian unfolding.*

Proof. In [ADL+08], it is shown that the band of any nested prismoid can be unfolded. That is, the band has at least one edge (not included in base and top) such that by cutting along the edge and unfolding continuously all faces of the band can be placed into a plane without intersection. Let the top and base polygons be $P_t = (a_1, a_2, \ldots, a_n)$ and $P_b = (b_1, b_2, \ldots, b_n)$, and suppose that the edge (a_1, b_1) allows us to unfold the band.

Then our Hamiltonian unfolding consists of $(b_{i+1}, b_{i+2}, \ldots, b_n, b_1, a_1, a_n, a_{n-1}, \ldots, a_3, a_2, b_2, b_3, \ldots, b_i)$

for some i with $i \geq 2$ (Figure 6). The index i is the first index such that the total turn angle from the vector $\overrightarrow{b_1 b_2}$ to the vector $\overrightarrow{b_i b_{i+1}}$ is greater than 90°. (Intuitively, the vertex b_{i+1} is the first vertex coming back to b_1. We note that i can be n.) We fix the base in the plane. Then the unfolding can be regarded as two "flipping" (Figure 7): one is the flipping of the top along the axis (b_1, b_2) with the trapezoid $a_1 a_2 b_2 b_1$ as a hinge, and the other one is the flipping of the band (except the trapezoid $a_1 a_2 b_2 b_1$) along the axis (b_i, b_{i+1}) with the trapezoid $a_i a_{i+1} b_{i+1} b_i$ as a hinge. Let $P'_t = (a'_1, a'_2, \ldots, a'_n)$ be the flipped top, and $Q = (b'_2, \ldots, b'_{i-1}, b_i, b_{i+1}, b'_{i+2} \ldots, b'_n, b'_1, a''_1, a''_n, \ldots, a''_3, a''_2)$ be the flipped band (except the trapezoid $b_1 b_2 a'_2 a'_1$). Let L_t and L_b be the line segments that contain $b_1 b_2$ and $b_i b_{i+1}$, respectively.

Now we prove that the Hamiltonian unfolding causes no overlap. We define the area A_t as the union of the rays ℓ perpendicular to L_t such that the endpoint of ℓ is on L_t and ℓ has a nonempty intersection with the flipped top (the left gray area in Figure 7). Let t_1 and t_2 be the rightmost and the leftmost points on L_t, respectively. For L_b and the flipped band, we also define A_b in a similar way. Let t_3 be the point on L_b closest to L_t. Then, it is easy to see that the flipped top is included in A_t and the flipped band is included in A_b.

We will show that A_b is above the line L_t, and hence A_t and A_b are separated by L_t. We have two cases. The first case is that the angle between the vector $\overrightarrow{b_1 b_2}$ and the vector $\overrightarrow{b_i b_{i+1}}$ is less than 180° as in Figure 7. This case is easy; the point t_3 closest to L_t is the intersection of L_b and the perpendicular to L_b that passes through b'_1 or b'_2. In the worst case, t_3 is at t_1. In this case, A_t and A_b have an intersection at this point, but this is the only point shared by A_t and A_b. Thus we can see that the Hamiltonian unfolding causes no overlap. Next we assume that the angle between the vector $\overrightarrow{b_1 b_2}$ and the vector $\overrightarrow{b_i b_{i+1}}$ is greater than 180°. In the case, we can use a symmetric argument at the point b_{i+1}. The worst case is that $b_{i+1} = b_n$ and t_3 is at t_2. Although A_t and A_b can have an intersection at this point, the Hamiltonian unfolding itself causes no overlap. □

3.2 General Prismoid

Theorem 5 *The number of Hamiltonian paths in a prismoid of $2n$ vertices is $n^3 + 2n^2$ for even n, and $n^3 + 2n^2 - n$ for odd n.*

Proof. Let $P_t = (a_1, a_2, \ldots, a_n)$ and $P_b = (b_1, b_2, \ldots, b_n)$ be the top and base polygons of the prismoid, respectively. We assume that a_i and b_i are joined by an edge for each $1 \leq i \leq n$. The key observation is that, once we add $(a_{i-1}, a_i, b_i, b_{i+1})$ or $(a_{i-1}, a_i, b_i, b_{i-1})$ as a subpath of a Hamiltonian path, the graph is separated into two parts at the edge $\{a_i, b_i\}$. Thus we have at most one consecutive zig-zag pattern $(a_{i-1}, a_i, b_i, b_{i+1}, a_{i+1}, a_{i+2}, b_{i+2}, \ldots)$ in a Hamiltonian path. The remaining part is filled by two paths in two different ways. The possible patterns are depicted in Figure 8 (the bold arrow indicates the start point of the zig-zag pattern from the vertex b_s). The first one (Figure 8(a)) divides the remaining part into two parts, say, the left and right part. Each of them is filled by a bending path. In the second one (Figure 8(b)), one of two subpaths spans the vertices in P_t, and the other subpath spans the vertices in P_b. (Thus the length of the zig-zag pattern is odd.)

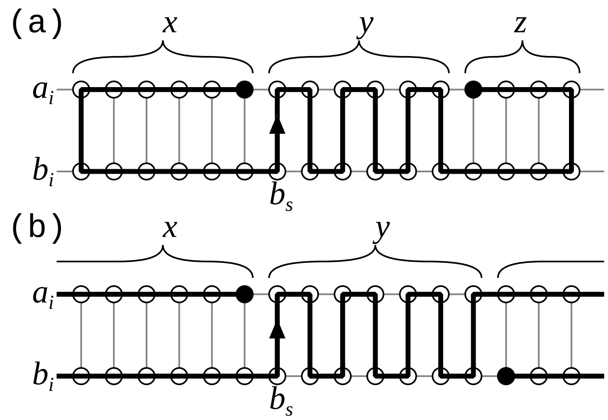

Figure 8: Two possible types of Hamiltonian paths in a prismoid

Now we count the number of possible Hamiltonian paths on the prismoid. We first assume that the unique zig-zag pattern starts from $(b_{s-1}, b_s, a_s, a_{s+1})$ as in Figure 8. Then the number of possible combinations of the first case (Figure 8(a)) is the number of partitions of n into three parts of size $x \geq 0$, $y \geq 0$, and $z \geq 0$ with $x + y + z = n$, which is equal to $\binom{n+1}{2}$. On the other hand, the number of possible combinations of the second case (Figure 8(b)) is the number of partitions of n into two parts of size $x \geq 0$ and (odd) $y \geq 0$ with $x + y = n$, which is equal to $\lfloor n/2 \rfloor$. Thus we have $\binom{n+1}{2} + \lfloor n/2 \rfloor$ Hamiltonian paths in the case. We have n ways to choose b_s, and we have the other case that the unique zig-zag pattern starts from $(a_{s-1}, a_s, b_s, b_{s+1})$. Therefore, we have $2n(\binom{n+1}{2} + \lfloor n/2 \rfloor)$ Hamiltonian paths on the prismoid. □

Corollary 6 *Hamiltonian-unfoldability of a prismoid can be determined in polynomial time. Moreover, all Hamiltonian-unfolding can be enumerated in polynomial time.*

Proof. We can check each cut along a Hamiltonian path in the prismoid to see if it gives us a nonoverlapping unfolding. By Theorem 5, the number of Hamiltonian paths in the prismoid is $O(n^3)$. Thus we can test all possible Hamiltonian unfoldings in polynomial time. $\square$

4 Conclusion

Some simple families of polyhedra that are edge-unfoldable were presented in [DO07]. Among them, it is easy to see that pyramids and prisms are also Hamiltonian-unfoldable, by so-called "band unfolding".

As we saw, any nested prismoid is Hamiltonian-unfoldable, and the Hamiltonian unfoldability of a general prismoid can be tested in polynomial time. We conjecture that all prismoids are Hamiltonian-unfoldable. It is worth mentioning that Aloupis showed in his thesis [Alo05] that the band of any prismoid (without top and bottom) can be unfolded. But a naive idea to attach the top and bottom to the unfolded band does not work; there are nested prismoids that cause overlap in any band unfolding [O'R12]. Since Hamiltonian-unfolding is more flexible than band unfolding, we may avoid overlapping for such prismoids.

A generalization of prismoids are *prismatoids*: a prismatoid is the convex hull of any two parallel convex polygons. Theorem 5 cannot be extended to prismatoids because some prismatoids have exponentially many Hamiltonian paths; see Figure 9.

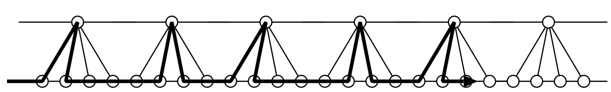

Figure 9: The side profile of a prismatoid that has exponentially many Hamiltonian paths

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